Journal of Approximation Theory **95**, 388–475 (1998)
Article No. AT973229

New Results on the Equilibrium Measure for Logarithmic Potentials in the Presence of an External Field

P. Deift

Courant Institute of Mathematical Sciences, New York, New York 10012

T. Kriecherbauer

Mathematisches Institut, Universität München, Munich, Germany

and

K. T.-R. McLaughlin

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

Communicated by Vilmos Totik

Received November 20, 1996; accepted in revised form December 8, 1997

In this paper we use techniques from the theory of ODEs and also from inverse scattering theory to obtain a variety of results on the regularity and support properties of the equilibrium measure for logarithmic potentials on the finite interval $[-1, 1]$, in the presence of an external field V. In particular, we show that if V is C^2, then the equilibrium measure is absolutely continuous with respect to Lebesgue measure, with a density which is Hölder-$\frac{1}{2}$ on $(-1, 1)$, and with at worst a square root singularity at ± 1. Moreover, if V is real analytic then the support of the equilibrium measure consists of a finite number of intervals. In the cases where $V = tx^m$, $m = 1, 2, 3$, or 4, the equilibrium measure is computed explicitly for all $t \in \mathbf{R}$. For these cases the support of the equilibrium measure consists of 1, 2, or 3 intervals, depending on t and m. We also present detailed results for the general monomial case $V = tx^m$, for all $m \in \mathbf{N}$.

The regularity results for the equilibrium measure are obtained by careful analysis of the Fekete points associated to the weight $e^{nV(x)}\,dx$. The results on the support of the equilibrium measure are obtained using two different approaches:

(i) an explicit formula of the kind derived by physicists for mean-field theory calculations;

(ii) detailed perturbation theoretic results of the kind that are needed to analyze the zero dispersion limit of the Korteweg–de Vries equation in Lax–Levermore theory.

The implications of the above results for a variety of related problems in approximation theory and the theory of orthogonal polynomials are also discussed.

0021-9045/98 $25.00

Contents

1. INTRODUCTION

It is a remarkable fact, arising from the work of many authors, in many different areas, and over many years, that a certain broad class of classical problems in analysis are intimately related. We begin by describing some of these problems and their interconnections, but first we need some notation and some definitions.

Let

$$\mathscr{A} = \left\{ \textit{Positive Borel measures } \mu, \int d\mu = 1, \textit{supp}(\mu) \subseteq \Sigma = [-1, 1] \right\}. \quad (1.1)$$

By an *external field* V we mean simply a map from Σ to $\mathbf{R}$. Henceforth, and throughout the paper, we assume that

$$\textit{all external fields } V \textit{ lie in } C^2(\Sigma), \quad (1.2)$$

i.e., $V \in C^2(-1, 1)$, and $V(x)$, $V'(x)$, and $V''(x)$ have continuous extensions to $[-1, 1]$. Although many of the definitions and results presented below are true for general closed subsets Σ of $\mathbf{R}^2$, and also for more general external fields V, we restrict our attention to the case $\Sigma = [-1, 1]$, with fields V that are twice continuously differentiable as in (1.2) above. Throughout the paper we use the following notation: if $S = \{x_1, ..., x_n\}$ is a set of n distinct points in $\mathbf{R}$, and δ_{x_i} is the Dirac delta measure concentrated at x_i, then

$$\rho_S = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \quad (1.3)$$

is the *normalized counting measure* for S.

Notational Remark. Note that in the orthogonal polynomial literature, the external field is usually denoted $-\frac{1}{2}V$, rather than V.

Problem 1. *Electrostatic Equilibrium*

Consider the maximization problem

$$E = E_V = \sup_{\mu \in \mathcal{A}} \left[\tfrac{1}{2} \iint \log(x-y)^2 \, d\mu(x) \, d\mu(y) + \int V(y) \, d\mu(y) \right]. \tag{1.4}$$

Here $\frac{1}{2} \iint \log(x-y)^2 \, d\mu(x) \, d\mu(y)$ is the equilibrium energy for a charge distribution $d\mu(x)$ on the *conductor* $\Sigma = [-1, 1]$, and $V \, d\mu$ is the energy of the charges in the external field $V(y)$. It is well known that the supremum in (1.4) is attained at a unique Borel measure $\mu = \mu_{ES}^V$ in $\mathcal{A}$; the measure μ_{ES}^V is called the *equilibrium measure* for Σ and V. Moreover, the equilibrium measure $d\mu_{ES}^V$ is characterized by the following variational conditions: for some constant ℓ,

$$V(x) + \int \log(x-y)^2 \, d\mu_{ES}^V(y) \leqslant l \tag{1.5}$$

with equality on the support of $d\mu_{ES}^V$.

Problem 2. *Weighted Transfinite Diameter*

For each positive integer n, let

$$d_{V,n} = \left[\max_{\{x_1, ..., x_n\} \subset [-1, 1]} \prod_{i<j} |x_j - x_i| \, e^{V(x_i)/2} e^{V(x_j)/2} \right]^{2/n(n-1)} \tag{1.6}$$

and set

$$d_V = \lim_{n \to \infty} d_{V,n}. \tag{1.7}$$

The quantity d_V is the *weighted transfinite diameter* for the interval $\Sigma = [-1, 1]$. Observe that in the unweighted case, $V = 0$, the quantity $d_{0,n}$ has the geometric interpretation as the maximum of the geometric means of the distances between n points located in Σ.

For each n, a set $\{x_1, ..., x_n\}$ which realizes the maximum of (1.6) is called an nth *weighted Fekete set*, and the points $x_1, ..., x_n$ are called *weighted Fekete points*. As is well known, Fekete points play a distinguished role in Lagrange interpolation. Also, in another direction, in the unweighted case and for a general closed set $\Sigma \subset \mathbf{R}^2$, Fekete [11] showed that the transfinite diameter d_0 controls the number N_Σ of polynomials with integer coefficients, and leading coefficient fixed, all of whose roots are simple and lie in Σ: if $d_0 < 1$, then $N_\Sigma < \infty$.

If $S^{(n)} = \{x_1^{(n)}, ..., x_n^{(n)}\}$ is an nth weighted Fekete set for $\Sigma = [-1, 1]$, consider

$$\mu_{TD}^V = \lim_{n \to \infty} \rho_{S^{(n)}}. \tag{1.8}$$

This limit is known to exist in the weak- $*$ topology for measures.

Problem 3. *Weighted Chebyshev Polynomials*

For each $n \geqslant 0$, set

$$T_{V,n} = inf \left\{ \sup_{|x| \leqslant 1} |e^{nV(x)/2} p(x) : \begin{array}{c} p(x) = x^n + \cdots \\ \textit{is a monic polynomial of degree } n \end{array} \right\}. \tag{1.9}$$

For each n, the infimum is attained at a unique polynomial, called the nth *weighted Chebyshev polynomial.* Furthermore,

$$T_v = \lim_{n \to \infty} (T_{V,n})^{1/n} \tag{1.10}$$

exists. Let $S^{(n)} = \{x_1^{(n)}, ..., x_n^{(n)}\}$ denote the zeros of the nth weighted Chebyshev polynomials, and consider

$$\mu_{WC}^V = \lim_{n \to \infty} \rho_{S^{(n)}}. \tag{1.11}$$

Once again, this limit is known to exist in the weak-$*$ topology of measures.

The Chebyshev polynomials, which play a distinguished role in approximation theory, are obtained as above by minimizing the supremum norm: to obtain the nth orthogonal polynomial corresponding to the measure $e^{nV(x)}\, dx$ on $[-1, 1]$, one must of course minimize the L^2 norm,

$$\left\{ \int |p(x)|^2 e^{nV(x)}\, dx : p(x) = x^n + \cdots \right\}.$$

Problem 4. *Zero Distribution of Orthogonal Polynomials*

Let $P(x) = x^m + \cdots$ be a monic polynomial of degree m. For $t > 0$, let $d\alpha_n^{t, \pm P}$ denote the measures $e^{\pm P(x)} X_{[-(tn)^{1/m}, (tn)^{1/m}]}\, dx$ obtained by restricting the measures $e^{\pm P(x)}\, dx$ to the interval $[-(tn)^{1/m}, (tn)^{1/m}]$. (Here X_B is the indicator function of the set B.) Let $p_n = p_n^{t, \pm P} = x^n + \cdots$ denote the nth monic orthogonal polynomial obtained by applying the Gram–Schmidt orthogonalization procedure to the sequence $1, x, ..., x^n$, with respect to the measure $d\alpha_n^{t, \pm P}$, in the usual way.

Let $y_1^{(n)}, ..., y_n^{(n)}$ denote the zeros of $p_n(x)$, and let $S^{(n)} = \{x_1^{(n)}, ..., x_n^{(n)}\}$, where $x_i^{(n)} = y_i^{(n)}/(tn)^{1/m}$ and consider the limit

$$\mu_{OP}^{t, \pm P} = \lim_{n \to \infty} \rho_{S^{(n)}}. \tag{1.12}$$

This limit is known to exist in the weak-$*$ topology, and in fact, is independent of the lower order terms of $P(x)$.

In the case where m is an even integer, the above problem arises naturally in the following way. Consider the nth monic orthogonal polynomial $\tilde{p}_n(x)$ obtained by applying the Gram–Schmidt procedure $1, x, ..., x^n$ with respect to the measure $e^{-P(x)}\,dx$ on $\mathbf{R}$.

QUESTION. What is the smallest interval $[-A, A]$ to which one can restrict $e^{-P(x)}\,dx$ in such a way that the nth polynomial p_n for the restricted measure $e^{-P(x)} X_{[-A, A]}(x)\,dx$ is a close approximation to $\tilde{p}_n$? See Remark (2) regarding Problem 4, below.

Problem 5. *Fast Decreasing Polynomials*

A set of polynomials $\{p_n(x), \textit{degree}(p_n) \leqslant n, n = 0, 1, ...\}$ is called a *set of fast decreasing polynomials* for a weight $e^{V/2}$, $V(0) = 0$, if there exists a constant C independent of n such that

$$p_n(0) = \quad \text{and} \quad |p_n(x)| \leqslant Ce^{-nV(x)/2} \quad \text{for } x \in [-1, 1] \quad \text{and} \quad n \geqslant 0. \tag{1.13}$$

Such polynomials arise in approximation theory, and provide a polynomial approximation to the Dirac delta function at the origin.

QUESTION. Given $V(x)$, do such polynomials exist?

Problem 6. *Hankel Determinants*

Let $d\alpha_n^V(x) = e^{nV(x)}\,dx$ be a measure on $[-1, 1]$. Let $c_j^{(n)} = \int x^j\,d\alpha_n^V(x)$, $j = 0, 1, ...$ denote the moments of $d\alpha_n^V(x)$, and for each $k \geqslant 1$, construct the *Hankel determinant*

$$H_k^{(n)} = \det(c_{i+j}^{(n)})_{0 \leqslant i, j \leqslant k}. \tag{1.14}$$

Consider

$$h_V = \lim_{n \to \infty} [H_n^{(n)}]^{1/n^2}. \tag{1.15}$$

This limit is known to exist.

Problems 1–6 are related in the following way.

Denote the support of the equilibrium measure $\mu = \mu_{ES}^V$ for the electrostatic maximization problem (1.4) by σ_{ES}^V. Then

$$\mu_{ES}^V = \mu_{TD}^V = \mu_{WC}^V \tag{1.16}$$

and in the case where $V(x) = \pm t x^m$, we have

$$\mu_{ES}^V = \mu_{TD}^V = \mu_{WC}^V = \mu_{OP}^{t, \pm P} \tag{1.17}$$

for any polynomial $P(x) = x^m + \cdots$. Furthermore, fast decreasing polynomials of type (1.13) exist if and only if

$$\text{equality holds in (1.5) at } x = 0. \tag{1.18}$$

Define the *weighted logarithmic capacity* of $\Sigma = [-1, 1]$ by

$$K = K_V = e^{E_V}, \tag{1.19}$$

where E_V is the supremum of the electrostatic energy in (1.4). Then, in addition,

$$K_V = d_V = T_V e^{\int V(y)/2 \, d\mu_{ES}^V(y)} = h_V. \tag{1.20}$$

It is clear from the above results and considerations that the key problem is to determine the equilibrium measure μ_{ES}^V. In fact, all that is needed is σ_{ES}^V, the support of μ_{ES}^V. Indeed, as we see below, μ_{ES}^V can be computed explicitly via the solution of a scalar Riemann–Hilbert problem, once σ_{ES}^V is known.

A general reference for results in potential theory, and in particular for the properties of the equilibrium measure of Problem 1 is [19]. The proof that the limit (1.7) exists in the unweighted case is due to Fekete [12]. The proof (in the unweighted case) that $d_V = K_V$ (see (1.20)) is due to Fekete [12] and Szegő [31]. The proof of the existence of the limit in (1.10), and the relation $K_V = T_V e^{\int V/2 \, d\mu_{ES}^V}$ (see (1.20)) is due to Gonchar and Rakhmanov [15], as are the proofs of the existence of the limit in (1.11) and the relation $\mu_{WC}^V = \mu_{ES}^V$ (see (1.16)). The proof that the limit in (1.12) exists, and that $\mu_{OP}^V = \mu_{ES}^V$ (see (1.17)) was given by Gonchar and Rakhmanov [15] (see also [23, 27] for the analogous problem on the whole line). Fast decreasing polynomials were considered, for example, by DeVore [7] and Nevai and Totik [25] (see also [28, 33]). The statement that fast decreasing polynomials exist if and only if (1.18) holds can be deduced from results in [28]. The proof that the limit in (1.15) exists and that $h_V = K_V$ can be deduced, for example, from results in [15].

An extremely useful reference containing all the above material, together with historical notes, is the recent book by Saff and Totik [28] (see also

[24]). Historically, it seems that Gauss was the first to begin the analysis of the electrostatic equilibrium problem with external fields. Working in 3 dimensions (so that the logarithmic potential is replaced by the $1/r$ Coulombic potential), Gauss [14] derived the Euler–Lagrange equations for the analog of (1.4), and solved these equations in simple cases. Gauss recognized early the analytical difficulties in computing the equilibrium measure for such problems, as he writes, [14, Section 35] "Die wirkliche Bestimmung der Vertheilung der Masse auf einer gegebenen Fläche für jede vorgeschriebene Form von U übersteigt in den meisten Fällen die Kräfte der Analyse in ihrem gegenwärtigen Zustande. (The actual determination of the distribution of the mass on a given surface for an arbitrary function U lies, in most cases, beyond the powers of present day analysis.)"

At this stage, what indeed is known about μ_{ES}^V and, in particular, σ_{ES}^V? It turns out that in recent years considerable progress on the determination of the general properties of σ_{ES}^V for one dimensional conductors Σ has come from yet another direction, viz., random matrix theory in the spirit of Wigner and Dyson (see [22]).

Consider, in particular, the space of $n \times n$ Hermitean matrices M with probability distribution

$$\Pi_n(M)\, dM = \tilde{Z}_n^{-1} e^{-n \operatorname{Tr}(V(M))}\, dM, \tag{1.21}$$

where $V(x) = a_{2m} x^{2m} + a_{2m-1} x^{2m-1} + \cdots + a_0$, $a_{2m} > 0$, and

$$dM = \left(\prod_{k=1}^{n} dM_{kk} \right) \prod_{k<j} d(ReM_{kj})\, d(ImM_{kj}) \tag{1.22}$$

is "Lebesgue" measure for Hermitean matrices, and $\tilde{Z}_n$ is the normalization factor. Using the spectral theorem $M = U \Lambda U^*$, $\Lambda = diag(\lambda_1(M), \ldots, \lambda_n(M))$, U unitary, as a change of variables, the distribution (1.21) takes the form (see [22])

$$\Pi_n(M)\, dM = Z_n^{-1} e^{-n \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{n} d\lambda_i\, dU, \tag{1.23}$$

where dU is the restriction to $U(n)/T(n)$, the quotient of the space of $n \times n$ unitary matrices by the n-torus, of Haar measure on $U(n)$, and Z_n is the normalization constant (partition function)

$$Z_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-n \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2\, d\lambda_1 \cdots d\lambda_n. \tag{1.24}$$

As is well known, a familiar calculation due to Heine (see [30]) shows that Z_n is precisely the Hankel determinant H_{n-1}^{n-1} which arises in the study of polynomials orthogonal with respect to the weight $d\alpha_n(x) = e^{-nV(x)}\,dx$ (see (1.14) above).

Let $N_n(\Delta)$ denote the integral of the normalized counting measure $1/n \sum_{i=1}^{n} \delta_{\lambda_i}$ for the eigenvalues $\lambda_1, ..., \lambda_n$ of a random Hermitean matrix M, over an interval $\Delta = (a, b)$: thus $N_n(\Delta) = n^{-1} \# \{\lambda_i \in \Delta\}$. Now consider $E(N_n(\Delta))$, the expectation of $N_n(\Delta)$ with respect to the probability measure (1.23). A simple calculation shows that

$$E(N_n(\Delta)) = \int_\Delta u_n(\lambda_1)\, d\lambda_1, \tag{1.25}$$

where $u_n(\lambda_1)$ is the $(n-1)$-fold integral,

$$u_n(\lambda_1) = Z_n^{-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-n \sum_{i=1}^{n} V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2\, d\lambda_2 \cdots d\lambda_n. \tag{1.26}$$

the so-called one point function for (1.23). The fact of the matter is the following: as $n \to \infty$, $E(N_n(\Delta))$ converges, and in addition to the connections (1.16)–(1.20) above, we have (see, in particular [26])

$$N(\Delta) \equiv \lim_{n \to \infty} E(N_n(\Delta)) = \mu_{ES}^{-V}(\Delta), \tag{1.27}$$

where μ_{ES}^{-V} is the equilibrium measure for $-V$, but now with conductor $\Sigma = (-\infty, \infty)$. Moreover the following is true.

Lemma 1.28 (See, for example, [26]). *N is absolutely continuous with respect to Lebesgue measure, $N(\Delta) = \int_\Delta \rho(\lambda)\, d\lambda$, with Radon–Nikodym derivative $\rho(\lambda)$, the so-called density of states, given by a Hölder continuous function with compact support consisting of a finite union of intervals.*

The proof of Lemma 1.28 follows from a remarkable identity for the Borel transform

$$U(z) = \int_{\mathbf{R}} \frac{N(d\lambda)}{\lambda - z}, \qquad z \notin \mathbf{R}, \tag{1.29}$$

of the limiting measure $N(\cdot)$. Indeed for $z \notin \mathbf{R}$, one has

$$U(z)^2 + V'(z)\, U(z) + L(z) = 0, \tag{1.30}$$

where

$$L(z) = \int_{\mathbf{R}} \frac{V'(z) - V'(\lambda)}{z - \lambda}\, N(d\lambda). \tag{1.31}$$

Solving for $U(z)$ from (1.30), and using well known properties of the Borel transform, the proof of (1.28) is immediate. Equation (1.30) was apparently first introduced by Bessis, Itzykson and Zuber in [2] in 1980, in the special case that V is quartic (see equation (A.4.35) ibid.). We refer the reader to [26] for a review of recent analytical results in random matrix theory, and also to the very interesting papers [3] and [16] and the many references therein.

Our goal in this paper is twofold. Firstly, we will prove a generalization of (1.28) which is appropriate for the finite conductor case, $\Sigma=[-1, 1]$, which allows V to be an arbitrary C^2 function, and which uses techniques that are perhaps more familiar to the orthogonal polynomial/approximation theory community. Indeed we will work directly with the weighted Fekete points arising in Problem 2 above. Secondly, we will analyze σ_{ES}^V in detail in the monomial case $V(x)=\pm tx^m$, $t>0$, both for m even and for m, odd, as t varies. Here we will use techniques introduced in [21] and [6] to analyze the continuum limit of the Toda lattice, and also the small dispersion limit of the Korteweg de Vries equation (see below).

In a sequel to this paper, we will show how to use the equilibrium measure μ_{ES}^V to compute the asymptotics of orthogonal polynomials via the steepest-descent/stationary-phase method for Riemann–Hilbert problems introduced in [9], and further developed in [10] and [8]. In related work, we refer the reader to the recent paper [1], in which the authors use Riemann–Hilbert techniques together with the theory of isomonodromy deformations to compute the asymptotics of orthogonal polynomials, and also to address the so-called "universality conjecture" arising in random matrix theory.

Our results are the following. Define

$$L_{1/2,\infty}=\{h(x)\ mble.\ on\ [-1,1]: \|h\|_{1/2,\infty}\equiv \sup_{|x|\leqslant 1}\sqrt{1-x^2}\,|h(x)|<\infty\}. \tag{1.32}$$

Define

$$q^{(0)}(x)=\left(\frac{V'(x)}{2}\right)^2+\int\frac{V'(x)-V'(y)}{x-y}\,d\mu_{ES}^V(y) + \frac{1}{x^2-1}\left(1+\int V'(y)(x+y)\,d\mu_{ES}^V(y)\right), \tag{1.33}$$

and let $q^{(0)}(x)=q_+^{(0)}(x)-q_-^{(0)}(x)$, $q_\pm^{(0)}\geqslant 0$, denote the decomposition of $q^{(0)}$ into positive and negative parts. As we will see in Section 2, the function $q^{(0)}$ arises naturally in the analysis of the weighted Fekete points $x_1^*, ..., x_n^*$.

THEOREM 1.34. *The equilibrium measure μ_{ES}^V for (1.4) with V in C^2 is absolutely continuous with respect to Lebesgue measure,*

$$d\mu_{ES}^V = \psi(x)\,dx, \qquad \psi(x) \geqslant 0, \tag{1.35}$$

where $\psi \in L_{1/2,\,\infty}$, and is given by

$$\psi = \frac{1}{\pi}\sqrt{q_-^{(0)}(x)} \leqslant \frac{c}{\sqrt{1-x^2}}, \tag{1.36}$$

for some $0 < c < \infty$. In particular ψ is continuous in $(-1, 1)$: if V is C^k, $k \geqslant 3$, then ψ is Hölder continuous of order $\frac{1}{2}$, but in general no better.

Inserting (1.35) into (1.33) above, we obtain

$$q^{(0)}(x) = \left(\frac{V'(x)}{2}\right)^2 + \int \frac{V'(x) - V'(y)}{x-y}\,\psi(y)\,dy + \frac{1}{x^2-1}\left(1 + \int V'(y)(x+y)\,\psi(y)\,dy\right). \tag{1.37}$$

Notational Remarks. For a function $F(z)$ defined in $\mathbf{C}\backslash\mathbf{R}$, we denote its boundary values on $\mathbf{R}$ by $F_\pm(z) = \lim_{\varepsilon \downarrow 0} F(z \pm i\varepsilon)$, $z \in \mathbf{R}$. There should be no confusion with the decomposition $q^{(0)} = q_+^{(0)} - q_-^{(0)}$ above.

THEOREM 1.38. *Suppose that $V(x)$ is real analytic in a neighborhood of $\Sigma = [-1, 1]$. Then in addition to the results of Theorem* 1.34, *ψ is supported on a* finite *number of subintervals in $[-1, 1]$.*

Suppose $supp(\mu_{ES}^V) = J = \bigcup_{j=0}^N (\alpha_j, \beta_j)$, where $-1 \leqslant \alpha_0 < \beta_0 < \alpha_1 \cdots < \alpha_N < \beta_N \leqslant 1$. There are five cases:

(i) $N = 0$, $\alpha_0 = -1$, $\beta_0 = 1$

(ii) $N > 0$, $\alpha_0 = -1$, $\beta_N = 1$

(iii) $N \geqslant 0$, $\alpha_0 = -1$, $\beta_N < 1$

(iv) $N \geqslant 0$, $\alpha_0 > -1$, $\beta_N = 1$

(v) $N \geqslant 0$, $\alpha_0 > -1$, $\beta_N < 1$

In case (i), *if $\int_{-1}^1 (iV'/(y^2-1)_+^{1/2})\,dy = 0$, then for $z \in \mathbf{C}\backslash\mathbf{R}$, define*

$$F(z) = [z^2-1]^{1/2}\,\frac{1}{\pi i}\int_{-1}^1 \frac{-iV'(y)/2\pi}{(y^2-1)_+^{1/2}}\,\frac{dy}{y-z} + \frac{i\gamma}{[z^2-1]^{1/2}}; \tag{1.39}$$

if $\int_{-1}^{1}(iV'/(y^2-1)_+^{1/2})\,dy\neq 0$, *then for* $z\in\mathbf{C}\backslash\mathbf{R}$, *define*

$$F(z)=\left[\frac{z+1}{z-1}\right]^{1/2}\frac{1}{\pi i}\int_{-1}^{1}\frac{-iV'(y)}{2\pi}\left(\frac{y-1}{y+1}\right)_+^{1/2}\frac{dy}{y-z}+\frac{i\gamma}{[z^2-1]^{1/2}}, \tag{1.40}$$

where in both (1.39) *and* (1.40), γ *is chosen so that* $\int_{-1}^{1} Re\,F_+(z)\,dz=1$. *In cases* (ii)–(v), *set*

$$F((z)=\frac{R(z)^{1/2}}{\pi i}\int_J\frac{-iV'(y)/2\pi}{R(y)_+^{1/2}}\frac{dy}{y-z},\qquad z\in\mathbf{C}\backslash\bar{J}, \tag{1.41}$$

where

$$R(z)=\left\{\begin{array}{ll}\dfrac{(z-\beta_0)(z-\alpha_N)}{(z^2-1)}\displaystyle\prod_{j=1}^{N-1}(z-\alpha_j)(z-\beta_j) & \textit{in case (ii)}\\ \dfrac{(z-\beta_0)}{(z+1)}\displaystyle\prod_{j=1}^{N}(z-\alpha_j)(z-\beta_j) & \textit{iń case (iii)}\\ \dfrac{(z-\alpha_N)}{(z-1)}\displaystyle\prod_{j=0}^{N-1}(z-\alpha_j)(z-\beta_j) & \textit{in case (iv)}\\ \displaystyle\prod_{j=0}^{N}(z-\alpha_j)(z-\beta_j) & \textit{in case (v)}\end{array}\right\}. \tag{1.42}$$

Then in all cases

$$\psi(x)=ReF_+(x), \tag{1.43}$$

and the support of μ_{ES}^V *is precisely the set* $\{x\colon ReF_+(x)>0\}$.

Notational Remark. It would be more usual for the closed set $\bar{J}=\bigcup_{j=0}^{N}[\alpha_j,\beta_j]$ to denote the support of μ_{ES}^V. As noted above, however, the open set $\{x\colon ReF_+(x)>0\}$ provides an effective description of the support: for this reason we use J,and not $\bar{J}$, to denote $supp(\mu_{ES}^V)$.

Remark 1. The radicals in Theorem 1.38 are chosen such that

$$(z^2-1)^{1/2}>0,$$
$$R(z)^{1/2}>0$$

for $z>\beta_N$. In (1.42), in case (ii) the product is taken to be 1 if $N=1$, and in cases (iii) and (iv), the products are taken to be 1 if $N=0$.

Remark 2. In [28] and [18] the authors identify classes of external fields which they prove lead to case (i), and case (v) with $N=0$, or case (ii) with $N=1$ and $\beta_0=-\alpha_1$.

Remark 3. In the calculations that follow it is useful to re-express the integrals in (1.39)–(1.41) by means of the residue theorem. Thus, for example, in cases (ii)–(v),

$$F(z) = -i\frac{V'(z)}{2\pi} + \frac{i}{2\pi} R(z)^{1/2} h(z), \tag{1.44}$$

where

$$h(z) = \frac{1}{2\pi i} \oint \frac{V'(y)}{R(y)^{1/2}} \frac{dy}{y-z} \tag{1.45}$$

and where the contour integral is taken in the counter clockwise direction around a closed loop which encircles the point z as well as $\bar{J}$. Clearly $h(z)$ is analytic in some open subset of $\mathbf{C}$ which contains the interval $[-1, 1]$. In particular, in the case where $V(x)$ is a polynomial, it follows by letting the contour go to infinity, that $h(z)$ is a polynomial.

THEOREM 1.46. *In the case that $V(x)$ is a polynomial of degree m, let the number of subintervals comprising supp(μ_{ES}^V) be given by $N+1 = N_V$. Then*

$$N_V \leqslant m+1. \tag{1.47}$$

As noted above, our final results provide more detailed information in the case that $V(x)$ is a monomial, $V(x) = \pm tx^m$, $t > 0$.

THEOREM 1.48. *If the external field is*

$$V(x) = -tx^{2q}, \qquad t > 0, \tag{1.49}$$

then there exists precisely one critical value $t_{-,2q}$, $t_{-,2q} = 1/q \prod_{\ell=1}^{q} 2\ell/(2\ell-1)$.

For $t_{-,2q} < t$, *the solution of the maximization problem falls into case* (v) *of Theorem* 1.38 *with* $\beta_0 = -\alpha_0$, *i.e., supp* $\mu_{ES}^V = (-\beta_0, \beta_0)$. *Moreover,* $\psi = ReF_+$, *with F defined in Theorem* 1.38, *case* (v), *and β_0 is determined by the equation*

$$\int_{-1}^{1} ReF_+(x)\, dx = 1, \tag{1.50}$$

or, equivalently,

$$B_0^{2q} tq \prod_{\ell=1}^{q} \frac{2\ell-1}{2\ell} = 1. \tag{1.51}$$

For $0 < t \leqslant t_{-,2q}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38 *above.*

Remark. *The above result is well known* (*see, e.g.,* [28]).

THEOREM 1.52. *Suppose the external field is*

$$V(x) = tx^{2q}, \qquad t > 0, \tag{1.53}$$

(**A**) *Suppose further that* $q = 1$. *Then there exists one critical value* $t_{+,2} = 2$.

A$_1$: $t_{+,2} < t$. *For* $t_{+,2} < t$, *the solution of the maximization problem falls into case* (ii) *of Theorem* 1.38, *with* $N = 1$, *and* $\beta_0 = -\alpha_1$, *i.e.*,

$$supp(\mu_{ES}^V) = (-1, -\alpha_1) \cup (\alpha_1, 1). \tag{1.54}$$

The parameter α_1, *and hence, through Theorem* 1.38, $\psi = ReF_+(x)$ *are determined by the equation*

$$\int_{-1}^{1} ReF_+(x)\, dx = 1. \tag{1.55}$$

As in Theorem 1.48 *above, this condition may be evaluated explicitly, and yields an algebraic equation for* α_1.

A$_2$: $0 < t \leqslant t_{+,2}$. *For* $0 < t \leqslant t_{+,2}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38, *i.e.*,

$$supp(\mu_{ES}^V) = (-1, 1). \tag{1.56}$$

(**B**) *Now suppose* $q = 2$. *Then there are two critical values* $t_{+,4}^{(2)} = 8/3 < t_{+,4}^{(1)}$.

B$_1$: $t_{+,4}^{(1)} \leqslant t$. *For* $t_{+,4}^{(1)} \leqslant t$, *the solution of the maximization problem falls into case* (ii) *of Theorem* 1.38, *with* $N = 1$, *as in part* (A) *above. That is,*

$$supp(\mu_{ES}^V) = (-1, -\alpha_1) \cup (\alpha_1, 1), \tag{1.57}$$

with α_1 *and* (*again*) $\psi = ReF_+(x)$ *determined by condition* (1.55), *with* F *defined in Theorem* 1.38, *with* $V(x) = tx^4$.

B$_2$: $t_{+,4}^{(2)} < t < t_{+,4}^{(1)}$. *For* $t_{+,4}^{(2)} < t < t_{+,4}^{(1)}$, *the solution of the maximization problem falls into case* (ii) *of Theorem* 1.38, *but now* $N = 2$. *More specifically, we have*

$$supp(\mu_{ES}^V) = (-1, -\alpha_2) \cup (-\beta_1, \beta_1) \cup (\alpha_2, 1). \tag{1.58}$$

The parameters β_1 *and* α_2 *are determined by the condition* (1.55) *together with a condition arising from the fact that for a variational problem such as* (1.4), *the Lagrange multiplier must be the same in all intervals comprising the complement of the support of* μ_{ES}^V (*see below*).

B$_3$ $0<t\leqslant t^{(2)}_{+,4}$. *For* $0<t\leqslant t^{(2)}_{+,4}$, *the solution of the maximization problem falls into case* (i) *of Theorem* 1.38. *That is,*

$$supp(\mu_{ES}^V)=(-1,1), \tag{1.59}$$

and $\psi(x)=ReF_+(x)$, *with* F *defined in Theorem* 1.38, *case* (i), *with* $V(x)=tx^4$.

(**C**) *Suppose* $q>2$. *Then there are two critical values* $t_{+,2q}<t^{(1)}_{+,2q}$ (*note there is no superscript on the first critical value*).

C$_1$: $t^{(1)}_{+,2q}\leqslant t$. *For* $t^{(1)}_{+,2q}\leqslant t$, *the maximization problem is described by case* (ii) *of Theorem* 1.38, *with* $N=1$.

C$_2$: $t^{(1)}_{+,2q}-\varepsilon<t<t^{(1)}_{+,2q}$. *For* $t^{(1)}_{+,2q}-\varepsilon<t<t^{(1)}_{+,2q}$, *the solution is described by case* (ii) *of Theorem* 1.38, *but again in this regime* $N=2$. *For a further description of the determination of the endpoints of the three intervals comprising* $supp(\mu_{ES}^V)$, *see below.*

C$_3$: $t_{+,2q}<t<t_{+,2q}+\varepsilon$. *For* $t_{+,2q}<t<t_{+,2q}+\varepsilon$, *the maximization problem is described by case* (ii) *of Theorem* 1.38, *with* $N=2$.

C$_4$: $0<t\leqslant t_{+,2q}$. *For* $0<t\leqslant t_{+,2q}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38.

Remark. Case (A) is essentially contained in [33].

Remark. Note that in the general case $q>2$, we do not have a global description of the maximizer ψ for all values of t. Of course, by Theorem 1.46, we know that the number of intervals is bounded by $m+1=2q+1$. We believe, however, that in the entire region $t_{+,2q}<t<t^{(1)}_{+,2q}$ $supp(\mu_{ES}^V)$ contains precisely three intervals.

Remark. Some of the quantities in Theorem 1.52 can easily be evaluated explicitly (e.g. $t_{+,2}=2$, $t_{+,4}=8/3$, $\alpha_1=(1-2/t)^{1/2}$ in case **A**$_1$), whereas other quantities are determined by more complicated relations. We have omitted most of this information in the statement of the Theorem, but it can be found in the proof of the Theorem below. This remark also applies to the following Theorem.

THEOREM 1.60. *Suppose the external field is*

$$V(x)=tx^{2q+1}. \tag{1.64}$$

(**A**) *Suppose further that* $q=0$. *Then there exists one critical value* $t_{+,1}=2$.

A$_1$: $t_{+,1}<t$. *For* $t_{+,1}<t$, *the solution of the maximization problem falls into case* (iv) *of Theorem* 1.38, *with* $N=0$, *i.e.*,

$$supp(\mu_{ES}^{V})=(\alpha_0,1). \tag{1.62}$$

A$_2$: $0<t\leqslant t_{+,1}$. *For* $0<t\leqslant t_{+,1}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38.

(**B**) *Now suppose* $q=1$. *Then there are three critical values* $t^{(3)}_{+,3}=4/3<t^{(2)}_{+,3}=25/6<t^{(1)}_{+,3}$.

B$_1$: $t^{(1)}_{+,3}\leqslant t$. *For* $t^{(1)}_{+,3}\leqslant t$, *the solution of the maximization problem falls into case* (iv) *of Theorem* 1.38, *with* $N=0$:

$$su\bar{p}p(\mu_{ES}^{V})=(\alpha_0,1). \tag{1.63}$$

B$_2$: $t^{(2)}_{+,3}<t<t^{(1)}_{+,3}$. *For* $t^{(2)}_{+,3}<t<t^{(1)}_{+,3}$, *the solution of the maximization problem falls into case* (iv) *of Theorem* 1.38, *but now* $N=1$. *More specifically, we have*

$$supp(\mu_{ES}^{V})=(\alpha_0,\beta_0)\cup(\alpha_1,1). \tag{1.64}$$

B$_3$: $t^{(3)}_{+,3}<t\leqslant t^{(2)}_{+,3}$. *For* $t^{(3)}_{+,3}<t\leqslant t^{(2)}_{+,3}$, *the solution of the maximization problem falls into case* (iv) *of Theorem* 1.38, *with* $N=0$. *That is*,

$$supp(\mu_{ES}^{V})=(\alpha_0,1). \tag{1.65}$$

B$_4$: $0<t\leqslant t^{(3)}_{+,3}$. *For* $0<t\leqslant t^{(3)}_{+,3}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38, *i.e.*,

$$supp(\mu_{ES}^{V})=(-1,1). \tag{1.66}$$

(**C**) *Suppose* $q>1$. *Then there are two critical values* $t_{+,2q+1}<t^{(1)}_{+,2q+1}$ (*note there is no superscript on the first critical value*).

C$_1$: $t^{(1)}_{+,2q+1}\leqslant t$. *For* $t^{(1)}_{+,2q+1}\leqslant t$, *the maximization problem is described by case* (iv) *of Theorem* 1.38, *with* $N=0$:

$$supp(\mu_{ES}^{V})=(\alpha_0,1). \tag{1.67}$$

C$_2$: $t^{(1)}_{+,2q+1}-\varepsilon<t<t^{(1)}_{+,2q+1}$. *For* $t^{(1)}_{+,2q+1}-\varepsilon<t<t^{(1)}_{+,2q+1}$, *the solution is described by case* (iv) *of Theorem* 1.38, *but again in this regime* $N=1$:

$$supp(\mu_{ES}^{V})=(\alpha_0,\beta_0)\cup(\alpha_1,1). \tag{1.68}$$

C$_3$: $t_{+,2q+1} < t < t_{+,2q+1} + \varepsilon$. *For* $t_{+,2q+1} < t < t_{+,2q+1} + \varepsilon$, *the maximization problem is described by case* (iv) *of Theorem* 1.38, *with* $N = 0$. *That is,*

$$supp(\mu_{ES}^{V}) = (\alpha_0, 1). \tag{1.69}$$

C$_4$: $0 < t \leqslant t_{+,2q+1}$. *For* $0 < t \leqslant t_{+,2q+1}$, *the solution of the maximization problem is described by case* (i) *of Theorem* 1.38, *i.e.*,

$$supp(\mu_{ES}^{V}) = (-1, 1). \tag{1.70}$$

Remark. As in Theorem 1.52 above, note that for general $q > 1$ we do not have a global description of the maximizer ψ for all values of $t > 0$. Again by Theorem 1.46, we know that the number of intervals is bounded by $m + 1 = 2q + 2$. However, we believe that for all $t \in (t_{+,2q+1}, t^{(1)}_{+,2q+1})$, the support of ψ contains precisely two intervals.

Remark. For Theorems 1.48, 1.52, 1.60, for t in any open interval (a, b) on which the number N_V is constant, the endpoints $\{\alpha_j\}$ and $\{\beta_j\}$ of the intervals comprising the support of μ_{ES}^{V} are analytic functions of t. On the other hand $J = \bigcup_{j=0}^{N} (\alpha_j, \beta_j)$ is continuous in the natural sense, even at the critical values of t, and hence μ_{ES}^{V} is weak-* continuous for all $t > 0$.

A picture of the support of μ_{ES}^{V} for these cases facilitates visualization of the results contained in Theorems 1.48–1.60. We fix m, and simultaneously plot all endpoints $\{\alpha_j(t)\}$ and $\{\beta_j(t)\}$ as functions of t, for $t > 0$. Connecting α_j and β_j with vertical lines at any fixed t, we have a snapshot of the support of μ_{ES}^{V}. As t varies, we see how the support evolves (see Figs. 1–7).

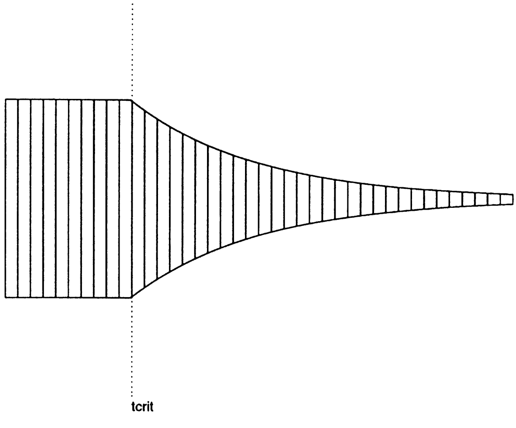

FIG. 1. Plot of the support J as a function of t, for the case $V = -tx^{2q}$, $t > 0$.

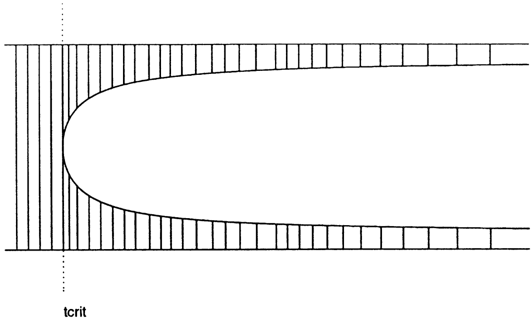

FIG. 2. Plot of the support J as a function of t, for the case $V = tx^2$, $t > 0$.

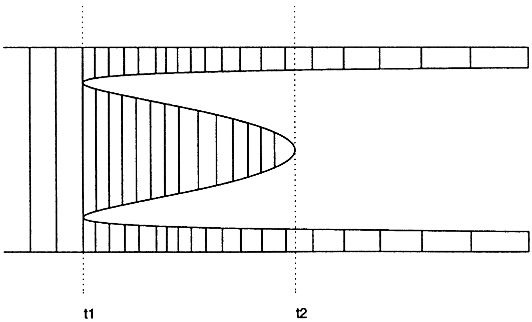

FIG. 3. Plot of the support J as a function of t, for the case $V = tx^4$, $t > 0$.

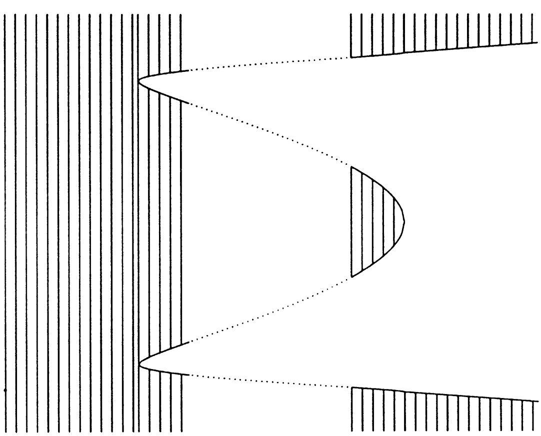

FIG. 4. Plot of the support J as a function of t, for the general case $V = tx^{2q}$, $t > 0$, $q \geqslant 3$. The dotted lines denote our conjecture for the region $t_{+,2q} + \varepsilon < t < t^{(1)}_{+,2q} - \varepsilon$.

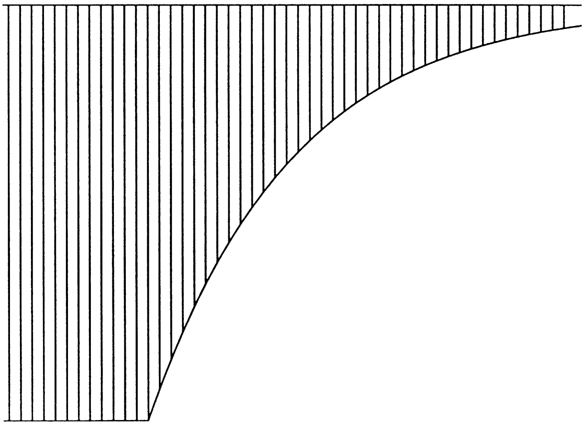

FIG. 5. Plot the support J as a function of t, for the case $V = tx$, $t > 0$.

Remark 1 (*Regarding Problem* 4). In Problem 4 above, it was mentioned that the limit in (1.12) is in fact independent of all lower order terms in the polynomial $P(x)$. Indeed, in [15], Gonchar and Rakhmanov show that

$$\mu_{OP}^{t, \pm P} = \mu_{ES}^{V} \tag{1.71}$$

with

$$V(x) = \pm tx^m, \tag{1.72}$$

where m is the degree of the weight-polynomial $P(x)$ in Problem 4. Thus Theorems 1.48–1.60 describe the asymptotic distribution of zeros of the polynomials of Problem 4.

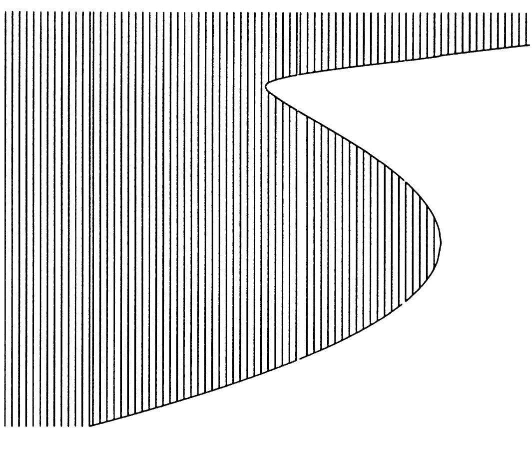

FIG. 6. Plot of the support J as a function of t, for the case $V = tx^3$, $t > 0$.

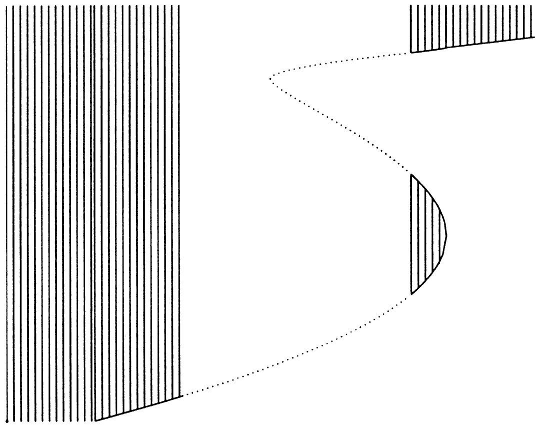

FIG. 7. Plot of the support J as a function of t, for the general case $V = tx^{2q+1}$, $t > 0$, $q \geqslant 2$. The dotted lines denote our conjecture for the region $t_{+,2q+1} + \varepsilon < t < t^{(1)}_{+,2q+1} - \varepsilon$.

Remark 2 (*Regarding Problem* 4). Rakhmanov [27] has shown that for measures of the form $e^{-tx^{2q}}\,dx$ on all of $\mathbf{R}$, the asymptotic distribution of zeros, when suitably scaled to the interval $[-1, 1]$, coincides with $\mu_{ES}^{-tx^{2q}}$, provided $t_{-,2q} \leqslant t$. Thus *the answer to the question posed at the end of* Problem 4 *is to take the "cutoff parameter"* $A = tn^{1/m}$, *with* $t \geqslant t_{-,2q}$.

Our approach to asymptotic problems for orthogonal polynomials is motivated by the work of Lax and Levermore [20], who considered the zero dispersion limit of the KdV equation with fixed initial data,

$$y_t - 6yy_x + \varepsilon^2 y_{xxx} = 0, \tag{1.73}$$

$$y(x, 0) = y_0(x), \tag{1.74}$$

as $\varepsilon \downarrow 0$. The authors used scattering and inverse scattering theory for the associated Lax operator

$$L = -\varepsilon^2 \frac{d^2}{dx^2} + y_0, \tag{1.75}$$

under the assumption that the effect of the reflection coefficient $r(z)$ is negligible, to obtain a formula in closed form for $y(x, t)$. In a remarkable calculation, they then showed that asymptotically as $\varepsilon \downarrow 0$ this formula is governed by an associated maximization problem, quite analogous to (1.4), but now with *two* external parameters x and t. They then proceeded to show that for all x and t this maximization problem possesses a unique solution, which is in fact an L^p function, for any $p \in [1, 2)$. Furthermore, they showed that at $t=0$, the support of the maximizer for each x is one interval, whose endpoints are related in an elegant way to the initial data $y_0(x)$. Until a critical later time t_b, the shock time for Burgers' equation $y_t - 6yy_x = 0$, $y(x, 0) = y_0(x)$, they showed that for all x the support of the maximizer is again precisely one interval, and that the relation between the endpoints and $y(x, t)$ which held at $t=0$ remains true for $0 \leqslant t \leqslant t_b$.

Beyond the critical time t_b, they postulated that for each x the support becomes a finite union of subintervals. However, this remained a conjecture until the work of Tian [32] (see also Wright [37]), who showed that for a very general class of initial data, as t crosses t_b, the support of the maximizer experiences a transition from one interval to two for values of x in an interval $(x_-(t), x_+(t))$; for $x > x_+(t)$ and $x < x_-(t)$, the maximizer continues to be supported on a single interval. While this result is local in t, Tian further showed that for a restricted class of initial data, the zero dispersion limit of the KdV equation is governed for all $t > t_b$ by a maximizer whose support for each x is either one interval,or two.

Our work uses an approach to the "phase transition" problem that is different from that of Tian's, and is based on the calculations and results [6, 8, 21], which in turn has provided a method to generalize the results of Tian (see [6]). Indeed,one can use the methods developed in Section 2 to prove the following result.

THEOREM 1.76. *For real analytic initial data $y_0(x)$, the number of intervals comprising the support of the maximizer remains finite for all x and t.*

We will provide a proof of this result in a later publication.

Lax–Levermore theory has been extended in a number of highly non-trivial directions, particularly in the work of Venakides [34–36]: whereas Lax and Levermore compute $y(x, t)$ to leading order as $\varepsilon \downarrow 0$, in [36] Venakides shows how to compute the asymptotics to the next order. This paper of Venakides in turn leads on to [8], which then provides the model for our approach to the computation of the asymptotics of orthogonal polynomials. As indicated above, our results for the asymptotics of orthogonal polynomials will be presented in a sequel to this paper.

The outline of the paper is as follows. In Section 2 we prove Theorem 1.34, deriving en route the advertised generalization of formula (1.30). Theorem 1.38 is proved in Section 3: this Section also contains an independent ode proof of the fact that $\psi(x)$ is supported on $\{x: q^{(0)}(x) < 0\}$. In Section 4 we prove Theorem 1.46, and also present some additional applications of Proposition 2.51. Section 5 is a summary of the variational conditions satisfied by the equilibrium measure. In Sections 6, 7, and 8, we prove Theorems 1.48, 1.52, and 1.60, respectively.

Remark. After completing this paper we learned that Kuijlaars and Dragnev [17] have solved the conjecture in the Remark immediately following Theorem 1.52, and Damelin and Kuijlaars [5] have solved the conjecture in the Remark immediately following Theorem 1.60. Subsequent to receiving these preprints, it became clear to the authors that the conjecture considered in [17] could also be proved directly using formula (1.37).

Remark. With more work using (1.37), one can obtain a sharper bound than Theorem 1.46, namely $N_V \leqslant (m/2)+1$ if m is even, and $N_V \leqslant (m+1)/2$ if m is odd. This is optimal in the sense that there exists polynomial external fields which achieve these bounds.

2. PROOF OF THEOREM 1.34—REGULARITY PROPERTIES OF THE EQUILIBRIUM MEASURE

We prove Theorem 1.34 by analyzing Problem 2 above. Thus, let

$$d_{V,n} = \left[\max_{\{x_1, \dots, x_n\} \subset [-1,1]} \prod_{i<j} |x_j - x_i| \, e^{V(x_i)/2} e^{V(x_j)/2} \right]^{2/n(n-1)} \tag{2.1}$$

as in (1.6), and in what follows, $\{x_1^*, \dots, x_n^*\}$, with $x_i^* < x_j^*$ for $i<j$, denotes an nth weighted Fekete set for this problem. Our first goal is to establish that then nearest neighbor distances $\{x_j^* - x_{j-1}^*\}$ are not too small as $n \to \infty$. We will show that this implies (1.35), and that the maximizer ψ satisfies $\psi \in L_{1/2,\infty}$.

Information about the weighted Fekete points is obtained through the analysis of the polynomial

$$f(x) = \prod_{j=1}^{n} (x - x_j^*). \tag{2.2}$$

It is a remarkable fact that for $V \in C^2$, $f(x)$ solves a linear second order differential equation with coefficients that are continuous in $(-1, 1)$. Our argument follows Szegő [30], where the equation is derived in special cases.

LEMMA 2.3. *The polynomial $f(x)$ defined in (2.2) above satisfies*

$$f'' + (n-1)\, V'(x)\, f'(x) = \frac{Q(x)}{x^2-1}\, f(x), \tag{2.4}$$

with

$$\begin{aligned} Q(x) &= n(n-1) + (n-1) \sum_{k=1}^{n} \frac{V'(x)(x^2-1) - V'(x_k^*)((x_k^*)^2-1)}{x - x_k^*} \\ &= n(n-1) + (n-1) \sum_{k=1}^{n} \int_0^1 \frac{d}{du} [\, V'(u)(u^2-1)] \Big|_{u = tx + (1-t)\, x_k^*} dt. \end{aligned} \tag{2.5}$$

Proof. The Fekete points $\{x_i^*\}$ satisfy one of the following:

(a) $-1 < x_1^*, \qquad x_n^* < 1,$

(b) $-1 = x_1^*, \qquad x_n^* < 1,$

(c) $-1 < x_1^*, \qquad x_n^* = 1,$

(d) $-1 = x_1^*, \qquad x_n^* = 1.$

In all cases, we have, from (2.1),

$$\frac{n-1}{2}\, V'(x_\ell^*) + \sum_{\substack{j=1 \\ j \neq l}}^{n} \frac{1}{x_\ell^* - x_j^*} = 0, \qquad \text{for} \quad \ell = 2, ..., n-1. \tag{2.6}$$

For $x \notin \{x_1^*, ..., x_n^*\}$, we have

$$f' = f \sum_{i=1}^{n} \frac{1}{x - x_i^*}, \tag{2.7}$$

and hence

$$f'' = f' \sum_{i=1}^{n} \frac{1}{x - x_i^*} - f \sum_{i=1}^{n} \frac{1}{(x - x_i^*)^2}. \tag{2.8}$$

Now

$$\frac{f''+(n-1)\,V'(x)\,f'}{f}$$

$$=\frac{f'}{f}\sum_{i=1}^{n}\frac{1}{x-x_i^*}-\sum_{i=1}^{n}\frac{1}{(x-x_i^*)^2}+(n-1)\,V'(x)\sum_{i=1}^{n}\frac{1}{x-x_i^*}$$

$$=\left(\sum_{i=1}^{n}\frac{1}{x-x_i^*}\right)^2-\sum_{i=1}^{n}\frac{1}{(x-x_i^*)^2}+(n-1)\sum_{i=1}^{n}\frac{V'(x)}{x-x_i^*}$$

$$=\sum_{j\neq k}\frac{1}{x-x_j^*}\frac{1}{x-x_k^*}+(n-1)\sum_{i=1}^{n}\frac{V'(x)}{x-x_i^*}$$

$$=\sum_{j\neq k}\left(\frac{1}{x-x_j^*}-\frac{1}{x-x_k^*}\right)\frac{1}{x_j^*-x_k^*}+(n-1)\sum_{i=1}^{n}\frac{V'(x)}{x-x_i^*}$$

$$=2\sum_{j\neq k}\frac{1}{x-x_j^*}\frac{1}{x_j^*-x_k^*}+(n-1)\sum_{i=1}^{n}\frac{V'(x)}{x-x_i^*}. \tag{2.9}$$

We define $\tilde{Q}$ by the relation

$$\tilde{Q}(x)=Q(x)-(n-1)\sum_{j=1}^{n}\frac{(x^2-1)\,V'(x)-((x_j^*)^2-1)\,V'(x_j^*)}{x-x_j^*}$$

$$=2(x^2-1)\sum_{j\neq k}\frac{1}{x-x_j^*}\frac{1}{x_j^*-x_k^*}$$

$$+(n-1)\sum_{j=1}^{n}\frac{((x_j^*)^2-1)\,V'(x_j^*)}{x-x_j^*},$$

where we have used that $Q=(x^2-1)((f''+(n-1)\,V'f')/f)$, and the last line of (2.9). The proof will be complete upon establishing that $\tilde{Q}=n(n-1)$. From (2.6), we have

$$\tilde{Q}=2(x^2-1)\sum_{j=1}^{n}\frac{1}{x-x_j^*}\sum_{k=1}^{n}{}'\frac{1}{x_j^*-x_k^*}$$

$$+\sum_{j=1}^{n}\left(\frac{(x_j^*)^2-1}{x-x_j^*}\right)\left[-2\sum_{k=1}^{n}{}'\frac{1}{x_j^*-x_k^*}\right] \tag{2.10}$$

(note that $(x_j^*)^2-1=0$ if $j=1$ and $x_1=-1$, or if $j=n$ and $x_n=1$. If not, then equation (2.6) is true also at $\ell=1$ and/or $\ell=n$). As usual $\sum'^{n}_{k=1} 1/(x_j^*-x_k^*)$ denotes the sum, with $k\neq j$, etc. We thus have

$$
\begin{aligned}
\tilde{Q} &= 2 \sum_{j=1}^{n} (x + x_j^*) \sum_{k=1}^{n}{}' \frac{1}{x_j^* - x_k^*} \\
&= 2 \sum_{j \neq k} \frac{x_j^*}{x_j^* - x_k^*} \\
&= \sum_{j \neq k} 1 = n(n-1).
\end{aligned} \tag{2.11}
$$

Thus equations (2.4), (2.5) are true for $x \notin \{x_1^*, ..., x_n^*\}$ if $V \in C^1$, and hence by continuity for all $x \in (-1, 1)$. ▌

It is useful to introduce

$$
g(x) = f(x)\, e^{(n-1)/2\, V(x)}. \tag{2.12}
$$

Note that of course the roots of g are precisely the weighted Fekete points. Furthermore, g satisfies

$$
g'' = q^{(n)} g, \tag{2.13}
$$

with

$$
q^{(n)}(x) = \frac{Q(x)}{x^2 - 1} + \left(\frac{n-1}{2} V'(x) \right)^2 + \frac{n-1}{2} V''(x). \tag{2.14}
$$

For future reference, note that Q depends on n.

In the following lemma we show that the weighted Fekete points may not be too close to each other.

Lemma 2.15. *The weighted Fekete points $\{x_i^*\}$ satisfy*

$$
x_{j+1}^* - x_j^* \geqslant \frac{C}{n} \sqrt{(1 - x_j^*)(1 - x_{j+1}^*)}, \qquad n \geqslant 1, \tag{2.16}
$$

for some constant $C > 0$, which depends only on V.

Proof. Note first that the potential $q^{(n)}$ satisfies

$$
|q^{(n)}| \leqslant \frac{\tilde{C} n^2}{1 - x^2}, \tag{2.17}
$$

This follows immediately from (2.14) and (2.5) above, and the fact that V is C^2.

To prove (2.16), we use the following inequalities satisfied by 2 consecutive roots x_j, x_{j+1} of any solution g of $-g'' + q^{(n)} g = 0$:

$$1 \leqslant \int_{x_j}^{x_{j+1}} (s - x_j)\, |q^{(n)}|\, ds \tag{2.18}$$

$$1 \leqslant \int_{x_j}^{x_{j+1}} (x_{j+1} - s)\, |q^{(n)}|\, ds. \tag{2.19}$$

The proof of (2.18) and (2.19) can be found, for example, in [4], problem number 3 of Chapter 9. We remark that although the potential $q^{(n)}$ considered here could possess singularities at ± 1, the fact that g is C^2 in $[-1, 1]$, up to the boundary, implies that the inequalities (2.18) and (2.19) hold for *all* roots of g including the case that g vanishes (and hence $g^{(n)}$ has a singularity) at -1 and/or $+1$.

From (2.17) and (2.18) we learn that

$$1 \leqslant \int_{x_j^*}^{x_{j+1}^*} (s - x_j^*)\, \frac{\tilde{C} n^2}{1 - s^2}\, ds. \tag{2.20}$$

As the integrand is a monotonically increasing function of s, the integral can be estimated as follows:

$$\int_{x_j^*}^{x_{j+1}^*} (s - x_j^*)\, \frac{\tilde{C} n^2}{1 - s^2}\, ds \leqslant (x_{j+1}^* - x_j^*)\, \frac{\tilde{C} n^2 (x_{j+1}^* - x_j^*)}{1 - (x_{j+1}^*)^2}. \tag{2.21}$$

Hence we have

$$x_{j+1}^* - x_j^* \geqslant \frac{\sqrt{(1 - x_{j+1}^*)(1 + x_{j+1}^*)}}{\sqrt{\tilde{C}}\, n}. \tag{2.22}$$

On the other hand, if we substitute (2.17) into (2.19), the integrand is now monotonically decreasing, and we obtain

$$x_{j+1}^* - x_j^* \geqslant \frac{\sqrt{(1 - x_j^*)(1 + x_j^*)}}{\sqrt{\tilde{C}}\, n}. \tag{2.23}$$

In order to complete the proof we distinguish the following cases:

(a) $x_{j+1}^* \leqslant 0$. Then we use (2.22) and $\sqrt{1 - x_{j+1}^*} \geqslant 1 \geqslant \sqrt{1 - x_j^*}/\sqrt{2}$.

(b) $x_j^* \geqslant 0$. Then we use (2.23) and $\sqrt{1 + x_j^*} \geqslant 1 \geqslant \sqrt{1 + x_{j+1}^*}/\sqrt{2}$.

(c) $x_j^* < 0 < x_{j+1}^*$: In case $x_{j+1}^* - x_j^* < \frac{1}{2}$, we have

$$\sqrt{1 - x_{j+1}^*} \geqslant \sqrt{\frac{1}{2}} \geqslant \frac{1}{\sqrt{3}} \sqrt{1 - x_j^*}, \tag{2.24}$$

and the result follows from (2.22); otherwise

$$x_{j+1}^* - x_j^* \geqslant \frac{1}{2} \geqslant \frac{\sqrt{(1-x_j^*)(1+x_{j+1}^*)}}{4}. \tag{2.25}$$

Defining $C = \min(1/\sqrt{3\tilde{C}}, 1/4)$, the desired result is established. ▮

We now show how the inequality (2.16) implies that the equilibrium measure μ_{ES}^V is absolutely continuous with respect to the Lebesgue measure, $d\mu_{ES}^V = \psi(x)\,dx$, and that the maximizer ψ lies in $L_{1/2,\infty}$.

Let $X = (C[-1, 1], \|\cdot\|_\infty)$, with its dual space denoted by X^*.

As above, we let $\rho_n = \rho_{S^{(n)}}$ denote the normalized counting measure corresponding to any nth Fekete set $S^{(n)} = \{x_j^*\}_{j=1}^n$. From the general results stated in the Introduction, in particular regarding Problem 2, we know that the sequence ρ_n converges in X^* to the equilibrium measure μ_{ES}^V.

Lemma 2.26. *The equilibrium measure μ_{ES}^V satisfies*

$$|\mu_{ES}^V(h)| \leqslant \frac{1}{C} \int_{-1}^{1} \frac{|h(x)|}{\sqrt{1-x^2}}\,dx, \qquad h \in X, \tag{2.27}$$

where C is the same constant as in Lemma 2.15.

Remark. Clearly this Lemma immediately implies that $d\mu_{ES}^V(x) = \psi(x)\,dx$, with $\psi(x) \in L_{1/2,\infty}$, and hence proves the first part of Theorem 1.34.

Proof. It suffices to prove that for $h \in X$ and $\eta > 0$ there is an integer $n_0 \in \mathbf{N}$ such that for all $n \geqslant n_0$,

$$|\rho_n(h)| \leqslant \frac{1}{C} \int_{-1}^{1} \frac{|h(x)|\,dx}{\sqrt{1-x^2}} + \eta. \tag{2.28}$$

Choose $0 < \varepsilon < 1$ such that $\|h\|_\infty / C\sqrt{\varepsilon} < \eta/16$. We decompose

$$\{1, ..., n\} = J_1 \cup J_2 \cup J_3 \cup J_4, \tag{2.29}$$

where

$$J_1 = \{1 \leqslant j \leqslant n : x_{j+1}^* \geqslant -1 + \varepsilon\} \tag{2.30}$$

$$J_2 = \{1 \leqslant j \leqslant n : x_j^* \leqslant -1 + \varepsilon < x_{j+1}^*\} \tag{2.31}$$

$$J_3 = \{1 \leqslant j \leqslant n : -1 + \varepsilon < x_j^* < 1 - \varepsilon\} \tag{2.32}$$

$$J_4 = \{1 \leqslant j \leqslant n : x_j^* \geqslant 1 - \varepsilon\}. \tag{2.33}$$

Note that J_2 contains at most 1 point.

Choose $n_0 = n_0(\varepsilon, \eta)$ such that the following two conditions hold:

I.

$$\frac{\|h\|_\infty}{n_0+1} < \frac{\eta}{8} \tag{2.34}$$

II. For every mesh $Z \subset [-1, +\varepsilon, 1-\varepsilon]$ with maximal width less than $4C/n_0$, one has

$$\left| R\left(\frac{|h|}{\sqrt{1-x^2}}, Z\right) - \int_{-1+\varepsilon}^{1-\varepsilon} \frac{|h|}{\sqrt{1-x^2}}\, dx \right| < \frac{\eta}{8}, \tag{2.35}$$

for any Riemann sum $R(g, Z)$ of the function $g = |h|/\sqrt{1-x^2}$ with respect to the mesh Z.

For $n > n_0$ we obtain the estimates

$$\begin{aligned} \frac{1}{n} \sum_{J_1} |h(x_j^*)| &\leqslant \frac{\|h\|_\infty}{n} \sum_{J_1} \frac{x_{j+1}^* - x_j^*}{x_{j+1}^* - x_j^*} \\ &\leqslant \frac{\|h\|_\infty}{C} \sum_{J_1} \frac{x_{j+1}^* - x_j^*}{\sqrt{1+x_{j+1}^*}} \\ &\leqslant \frac{\|h\|_\infty}{C} \int_{-1}^{-1+\varepsilon} \frac{dx}{\sqrt{1+x}} \\ &= 2\frac{\|h\|_\infty}{\mathrm{C}} \sqrt{\varepsilon} < \frac{\eta}{8}, \end{aligned} \tag{2.23}$$

where the second inequality follows from Lemma 2.15. Furthermore, we also have

$$\frac{1}{n} \sum_{J_2} |h(x_j^*)| \leqslant \frac{\|h\|_\infty}{n} < \frac{\eta}{8}, \tag{2.37}$$

where the inequalities in (2.37) follow from condition I, and the fact that J_2 contains at most one integer.

To bound $\sum_{J_3} |h(x_j^*)|$, we proceed as follows. Define a mesh $Z = \{z_j \colon j \in \tilde{J}_3\}$ on $[-1+\varepsilon, 1-\varepsilon]$ by adding points $\{y_j^{(i)}\}$ to $\{x_j^* \colon j \in J_3\}$ in the following way: if $x_{j+1}^* \leqslant 1-\varepsilon$, and $x_{j+1}^* - x_j^*$ is bigger than $4C/n_0$ we insert points $x_j^* = y_j^{(0)} < y_j^{(1)} < \cdots < y_j^{(\ell)} < y_j^{(\ell+1)} = x_{j+1}^*$ such that

$$\frac{4C}{n_0} \geqslant y_{j+1}^{(i+1)} - y_j^{(i)} \geqslant \frac{2C}{n_0} \qquad \text{for all} \quad 0 \leqslant i \leqslant \ell. \tag{2.38}$$

Notice that

$$\frac{2C}{n_0} \geqslant \frac{C}{n_0} \sqrt{(1-y_j^{(i)})(1+y_j^{(i+1)})} \geqslant \frac{C}{n} \sqrt{(1-{}_j^{(i)})(1+y_j^{(i+1)})} \qquad (2.39)$$

for $0 \leqslant i \leqslant \ell$ and therefore by Lemma 2.15, we have

$$|z_{j+1} - z_j| \geqslant \frac{C}{n} \sqrt{(1-z_j)(1+z_{j+1})} \qquad (2.40)$$

for all $j \in \tilde{J}_3$. Note that there is at most 1 point $x_j^* = x_{j_0}^*$, $j \in J_3$, such that $x_{j_0+1}^* \geqslant 1-\varepsilon$; let $\hat{J}_3$ denote $\tilde{J}_3 \backslash \{j_0^*\}$.

Then we have

$$\begin{aligned} \frac{1}{n} \sum_{j \in J_3} |h(x_j^*)| &\leqslant \frac{1}{n} \sum_{j \in \hat{J}_3} |h(z_j)| + \frac{|h(x_{j_0}^*)|}{n} \\ &= \frac{1}{n} \sum_{j \in \hat{J}_3} |h(z_j)| \frac{\Delta z_j}{\Delta z_j} + \frac{\|h\|}{n} \\ &\leqslant \frac{1}{C} \sum_{\hat{J}_3} \frac{|h(z_j)| \, \Delta z_j}{\sqrt{(1-z_j)(1+z_{j+1})}} + \frac{\|h\|}{n} \\ &\leqslant \frac{1}{C} \sum_{\hat{J}_3} \frac{|h(z_j)|}{\sqrt{1-z_j^2}} \Delta z_j + \frac{\|h\|}{n} \\ &\leqslant \frac{1}{C} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{|h(y)|}{\sqrt{1-y^2}} \, dy + \frac{\eta}{4}, \end{aligned} \qquad (2.41)$$

where we have used (2.40) and II above. Lastly, as in (2.36) above, but now treating x_n^* separately, we have

$$\frac{1}{n} \sum_{j \in J_4} |h(x_j^*)| \leqslant \frac{1}{n} \sum_{\substack{j \in J_4 \\ j \neq n}} |h(x_j^*)| + \frac{|h(x_n^*)|}{n} \leqslant \frac{\eta}{4}. \qquad (2.42)$$

Combining (2.36), (2.37), (2.41), and (2.42), we have established that for each $h \in X$, and for each $\eta > 0$, there exists $n_0 \in \mathbf{N}$ such that for all $n \geqslant n_0$ we have

$$|\rho_{S^{(n)}}(h)| \leqslant \frac{1}{C} \int_{-1}^{1} \frac{|h|}{\sqrt{1-x^2}} \, dx + \frac{3\eta}{4}. \qquad (2.43)$$

This completes the proof of the Lemma. ▮

We now prove the generalization of (1.30) advertised above for external fields which are C^2 on $[-1, 1]$. As we will see, an immediate consequence of (1.30) is the formula $\psi(x) = 1/\pi \sqrt{q_-^{(0)}(x)}$, which together with (1.33) completes the proof of Theorem 1.34.

As $d\mu_{ES}^V(x) = \psi(x)\,dx$ is absolutely continuous, the variational conditions (1.5) in Problem 1 imply the following result. Let

$$H(h) = \frac{1}{\pi} ⨍ \frac{h(y)}{x-y}\,dy \tag{2.44}$$

denote the Hilbert transform of a function $h \in L^p$ for some $p > 1$ (as usual, $⨍$ denotes the Cauchy principle value). Recall that H is a bounded map from L^p to L^p, for $p > 1$ (see [29], for example, for the properties of H).

LEMMA 2.45. *Let $d\mu_{ES}^V(x) = \psi(x)\,dx$, $\psi \in L_{1/2,\,\infty}$ be the unique maximizer in* (1.4). *Then*

$$H\psi(x) = -\frac{V'(x)}{2\pi} \tag{2.46}$$

for a.e. $x \in \{y: \psi(y) > 0\}$.

Proof. For almost all $x \in \{y: \psi(y) > 0\}$, we have

$$\int \log(x-y)^2\,\psi(y)\,dy + V(x) = \ell. \tag{2.47}$$

However, as ψ clearly belongs to L^p for any $1 \leqslant p < 2$, $H\psi \in L^p$, and by the fundamental theorem of calculus, we obtain

$$\int \log(x-y)^2\,\psi(y)\,dy = 2\pi \int_0^x H\psi(y)\,dy + c \tag{2.48}$$

for some constant c and for *all* $x \in \mathbf{R}$. Thus

$$2\pi \int_0^x H\psi(y)\,dy + V(x) = \ell - c \tag{2.49}$$

almost everywhere on $\{y: \psi(y) > 0\}$. Using the fact that almost every point of this set is a point of density, equation (2.49), together with the fundamental theorem of calculus, implies (2.46). ∎

We define

$$F(z) = \frac{1}{\pi i} \int_{-1}^{1} \frac{\psi(y)\,dy}{y-z}, \qquad z \in \mathbf{C} \setminus [-1, 1]. \tag{2.50}$$

Recall (see, e.g., [29], III) that for almost all $x \in (-1, 1)$, $F_{\pm}(x) = \lim_{\varepsilon \downarrow 0} F(x \pm i\varepsilon)$ exists and equals $\pm\psi(x) + iH\psi(x)$.

PROPOSITION 2.51. *Let $q^{(0)}(x)$ be the function defined by* (1.37). *Then*

$$\left(F_{\pm}(x) + \frac{iV'(x)}{2\pi}\right)^2 + \frac{q^{(0)}(x)}{\pi^2} = 0 \qquad \textit{for a.e.} \quad x \in (-1, 1), \tag{2.52}$$

and

$$\frac{q^{(0)}(x)}{\pi^2} = \left(H\psi(x) + \frac{V'(x)}{2\pi}\right)^2 - \psi(x)^2 \qquad \textit{a.e.} \quad x \in (-1, 1). \tag{2.53}$$

Furthermore, (2.53) *is the decomposition of*

$$\frac{q^{(0)}(x)}{\pi^2} = \frac{1}{\pi^2}\,(q_{+}^{(0)}(x) - q_{-}^{(0)}(x)), \qquad q_{\pm}^{(0)}(x) \geqslant 0,$$

into positive and negative parts. In particular, we have

$$\psi(x) = \frac{1}{\pi}\sqrt{q_{-}^{(0)}(x)} \qquad \textit{for all} \quad x \in (-1, 1). \tag{2.54}$$

Proof. For $x \notin \Sigma = [-1, 1]$, consider

$$h(x) = F^2(x) + \frac{1}{\pi^2}\int_{-1}^{1} \frac{V'(t)\,\psi(t)\,dt}{t - x}. \tag{2.55}$$

Clearly $h(x)$ is analytic in $\mathbf{C}\backslash\Sigma$. Also for almost all $x \in (-1, 1)$,

$$\begin{aligned} h_{\pm}(x) &= F_{\pm}^2(x) + \frac{i}{\pi}\,(\pm V'\psi + iH(V'\psi)) \\ &= (\pm\psi + iH\psi)^2 \pm \frac{i}{\pi}\,V'\psi - \frac{H(V'\psi)}{\pi} \\ &= \psi^2 - (H\psi)^2 \pm 2i\psi H\psi \pm \frac{i}{\pi}\,V'\psi - \frac{H(V'\psi)}{\pi}. \end{aligned} \tag{2.56}$$

But for $\psi(x) > 0$, $H\psi(x) = -V'(x)/2\pi$, by Lemma 2.45. Thus

$$(\psi(H\psi))(x) = -\psi(x)\,\frac{V'(x)}{2\pi} \qquad \text{for a.e.} \quad x \in (-1, 1). \tag{2.57}$$

This implies that

$$h_{\pm}(x) = \psi^2 - (H\psi)^2 - \frac{1}{\pi} H(V'\psi) \pm 2i\left(-\frac{V'}{2\pi}\right)\psi \pm \frac{i}{\pi} V'\psi$$
$$= \psi^2 - (H\psi)^2 - \frac{1}{\pi} H(V'\psi), \tag{2.58}$$

and hence

$$h_+(x) = h_-(x) \qquad \text{a.e. on } (-1, 1). \tag{2.59}$$

As $\psi \in L_{1/2, \infty}$, it follows that $h_{\pm}(x) \in L^p(a, b)$ for any subinterval $[a, b] \subset (-1, 1)$ and for some (and in fact all) $1 < p < \infty$. A standard argument using Cauchy's integral formula and (2.59) then implies that h is analytic across $(-1, 1)$. Hence $h(x)$ is analytic in $\mathbf{C}\backslash\{-1, 1\}$. A simple argument using the fact that $\psi \in L_{1/2, \infty}$ then shows that $h(x)$ has at worst simple poles at ± 1. Thus we learn that $(x^2-1)\, h(x)$ is entire.

Evaluating as $x \to \infty$, we find

$$(x^2-1)\, h(x)$$
$$= (x^2-1)\left(-\frac{1}{\pi^2 x^2} - \frac{1}{\pi^2 x}\int V'\psi\, dt - \frac{1}{\pi^2 x^2}\int tV'\psi\, dt + O\left(\frac{1}{x^3}\right)\right)$$
$$= -\frac{1}{\pi^2} - \frac{x}{\pi^2}\int {}'\psi\, dt - \frac{1}{\pi^2}\int tV'\psi\, dt + O\left(\frac{1}{x}\right), \tag{2.60}$$

and hence for $x \notin [-1, 1]$,

$$F^2(x) + \frac{1}{\pi^2}\int \frac{V'\psi\, dt}{t-x} + \frac{1}{\pi^2(x^2-1)}\left(1 + \int (x+t)\, V'\psi\, dt\right) = 0, \tag{2.61}$$

which implies, in particular, for almost all x,

$$F_{\pm}^2(x) + \frac{i}{\pi}(\pm V'\psi + iH(V'\psi))$$
$$+ \frac{1}{\pi^2(x^2-1)}\left(1 + \int (x+t)\, V'\psi\, dt\right) = 0. \tag{2.62}$$

With some simple algebra, we immediately obtain (2.52); together with (2.57), this in turn implies (2.53), and hence (2.54). ∎

Observe that if $V(x)$ is real analytic in a neighborhood D of $[-1, 1]$, then simple algebra applied to equation (2.61) shows that for $x \in D \setminus [-1, 1]$,

$$F^2(x) + \frac{i}{\pi} V'(x) F(x) + T(x) = 0, \tag{2.63}$$

where

$$T(x) = \frac{1}{\pi^2} \int \frac{V'(x) - V'(t)}{x - t} \psi(t)\, dt + \frac{1}{\pi^2 (x^2 - 1)} \left(1 + \int (t + x) V'(t) \psi(t)\, dt \right) \tag{2.64}$$

Formula (2.63) is the generalization of (1.30) promised in the introduction. (Set $U(x) = i\pi F(x)$, and note that V in (1.30) is the negative of the external field that we use in this paper.)

Relation (2.61) is the limiting form of the Ricatti equation associated with (2.4) in the standard way. In the case where V is real analytic in a neighborhood D of $[-1, 1]$ say, this can be seen as follows.

Using the fact that

$$\frac{f'}{f} = \sum_{i=1}^{n} \frac{1}{x - x_i^*} \tag{2.65}$$

for $x \notin \{x_1^*, ..., x_n^*\}$, we obtain from equation (2.4)

$$\left(\sum_{i=1}^{n} \frac{1}{x - x_i^*} \right)' = -(n-1) V'(x) \left(\sum_{i=1}^{n} \frac{1}{x - x_i^*} \right) + \frac{Q(x)}{x^2 - 1} - \left(\sum_{i=1}^{n} \frac{1}{x - x_i^*} \right)^2.$$

By analytic continuation, this relation remains true for all $x \in D \setminus \{-1, x_1, ..., x_n, 1\}$. In particular, for $x \in D \setminus [-1, 1]$, we have

$$\begin{aligned} \frac{1}{n} \left(\int \frac{1}{x - y}\, d\rho_n(y) \right)' &= -\frac{(n-1)}{n} V'(x) \left(\int \frac{1}{x - y}\, d\rho_n(y) \right) + \frac{n-1}{n} \int \frac{V'(x) - V'(y)}{x - y}\, d\rho_n(y) \\ &\quad + \frac{n-1}{n(x^2 - 1)} \left(1 + \int (y + x) V'(y)\, d\rho_n(y) \right) - \left(\int \frac{1}{x - y}\, d\rho_n(y) \right)^2. \end{aligned} \tag{2.66}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} 0 = & -V'(x)\left(\int \frac{1}{x-y}\,\psi(y)\,dy\right) \\ & + \int \frac{V'(x)-V'(y)}{x-y}\,\psi(y)\,dy \\ & + \frac{1}{(x^2-1)}\left(1+\int (y+x)\,V'(y)\,\psi(y)\,dy\right) \\ & - \left(\int \frac{1}{x-y}\,\psi(y)\,dy\right)^2 \end{aligned} \tag{2.67}$$

for $x \in D \backslash [-1, 1]$. Simple algebra now leads to (2.61).

Also, observe that the fundamental relation $\psi(x) = 1/\pi \sqrt{q_-^{(0)}(x)}$ implies the interesting fact that the function $q_-^{(0)}(x)$ is not identically 0; indeed $1/\pi \int_{-1}^{1} \sqrt{q_-^{(0)}(x)}\, dx = 1$. Although $q_-^{(0)}(x)$ depends on $d\mu_{ES}^V = \psi(x)\, dx$, the relation (2.54) can still be used to obtain valuable information about $\psi(x)$. Indeed, this is the thrust of Theorem 1.38.

Notational Remark. In the proofs that follow we will often omit the signature on square roots that appear in the formulae below. For example, in (3.7) below, we write

$$\int \frac{-iV'}{2\pi R(y)^{1/2}} \frac{dy}{y-z},$$

where we should properly write

$$\int \frac{-iV'}{2\pi R(y)_+^{1/2}} \frac{dy}{y-z}.$$

We adopt the following convention: whenever there is any ambiguity in a square root appearing in a formula, we always take the value of the root from above, i.e, $(\cdot)^{1/2} = (\cdot)_+^{1/2}$.

3. PROOF OF THEOREM 1.38—REPRESENTATION OF THE EQUILIBRIUM MEASURE IN THE ANALYTIC CASE

As $V'(x)$ is real analytic in a neighborhood D of $[-1, 1]$, it follows from (1.37) that $q^{(0)}(x)$ is real analytic in $D \backslash \{\pm 1\}$, with polar singularities at ± 1. On the other hand, as observed above, $q^{(0)}(x) \not\equiv 0$, and hence $q^{(0)}(x) < 0$ on a finite number of subintervals $J = \bigcup_{j=0}^{N} (\alpha_j, \beta_j)$ in $[-1, 1]$, $-1 \leqslant \alpha_0 < \beta_0 < \cdots < \alpha_N < \beta_N \leqslant 1$.

We now consider the integral representations (1.39)–(1.43). We give the proof only in case (ii); the remaining cases are similar.

It follows from Lemma 2.45 that

$$F(z)=\frac{1}{\pi i}\int \frac{\psi(t)\,dt}{t-z} \tag{3.1}$$

satisfies

$$F_+(x)+F_-(x)=2i(H\psi)(x)=-\frac{iV'(x)}{\pi}, \qquad \text{a.e.} \quad x\in J, \tag{3.2}$$

$$F_+(x)-F_-(x)=0, \qquad x\in[-1,1]\setminus\bar{J}. \tag{3.3}$$

Furthermore, we have the following result.

Property 3.4. $(z^2-1)\,F(z)$ has $L^p_{loc}(\mathbf{R})$ boundary values for all $1<p<\infty$.

Property (3.4) follows by a simple argument using the fact that $\psi\in L_{1/2,\,\infty}$ (c.f. paragraph after (2.59)).

Let

$$R=\frac{(z-\beta_0)(z-\alpha_N)}{(z^2-1)}\prod_{j=1}^{N-1}(z-\alpha_j)(z-\beta_j)$$

as in (1.42), and consider

$$g(z)=\frac{F(z)}{R(z)^{1/2}}-\frac{1}{\pi i}\int_J -\frac{iV'}{2\pi R(y)^{1/2}}\frac{dy}{y-z}, \tag{3.5}$$

with F defined in (3.1). Direct computation, together with (3.2) and (3.3), shows that

$$g_+(x)-g_-(x)=0 \qquad \text{for} \quad x\in\mathbf{R}\setminus\{\alpha_0=-1,\alpha_1,\ldots,\alpha_N,\beta_0,\ldots,\beta_N=1\}, \tag{3.6}$$

and we conclude that g is analytic in $\mathbf{C}\setminus\{\alpha_0=-1,\alpha_1,\ldots,\alpha_N,\beta_0,\ldots,\beta_N=1\}$. Now

$$(z^2-1)\,g(z)=\frac{1}{R(z)^{1/2}}\left[(z^2-1)\,F(z)-\frac{(z^2-1)\,R^{1/2}}{\pi i}\times\int_J -\frac{iV'}{2\pi R(y)^{1/2}}\frac{dy}{y-z}\right]. \tag{3.7}$$

But, from (1.44)–(1.45), we see that

$$(z^2-1)\, R^{1/2}\,\frac{1}{\pi i}\int_J \frac{-iV'}{2\pi R(y)^{1/2}}\,\frac{dy}{y-z}$$

is continuous down to the axis, and hence $G(z)=R^{1/2}(z^2-1)\,g(z)$ has boundary values in $L^p_{loc}(\mathbf{R})$ for any $1<p<\infty$. It follows that $(z^2-1)\,g(z)=G/R^{1/2}$ has, in particular, boundary values in L^1, and hence the possible singularities of $(z^2-1)\,g(z)$ at the endpoints of J are removeable. In particular, near $z=1$, we have

$$(z^2-1)\,g(z)=a_0+b_0(z-1)\cdots. \tag{3.8}$$

But using the fact that

$$(z^2-1)\,\frac{R^{1/2}}{\pi i}\int_J \frac{-iV'}{2\pi R(y)^{1/2}}\,\frac{dy}{y-z}$$

is continuous in a neighborhood of $z=1$, and also the fact that $\psi\in L_{1/2,\,\infty}$, it is easy to see (cf. again paragraph following (2.59)) that $\lim_{z\downarrow 1}(z^2-1)\,g(z)=0$, and hence $a_0=0$. It follows that $g(z)$ is analytic near $z=1$, and by a similar argument, near $z=-1$. The above considerations show that $g(z)$ is entire. But, as z goes to ∞, it is clear from (3.5) that $g(z)\to 0$. Hence $g(z)\equiv 0$, and this proves formulae (1.39)–(1.43) in case (ii).

Remark. As $F(z)=1/\pi iz+O(1/z^2)$, we obtain a set of moment conditions

$$\int_J \frac{-iV'(y)}{2\pi R(y)^{1/2}}\,y^j\,dy=\delta_{j,\,N-1}, \qquad 0\leqslant j\leqslant N-1, \tag{3.9}$$

satisfied by the endpoints of the set J. To complete these equations to a full set of $2N$ equations (recall that in this case $\alpha_0=-1$ and $\beta_N=1$ are known), we require that

$$\int_{\beta_j}^{\alpha_{j+1}}\left(H\psi+\frac{V'}{2\pi}\right)dy=0, \qquad 0\leqslant j\leqslant N-1. \tag{3.10}$$

Equation (3.10) follows from the requirement that

$$\int \log(x-y)^2\,\psi(y)\,dy+V(x)=\ell \qquad \text{for all} \quad x\in J. \tag{3.11}$$

(Recall equation (2.48), $L\psi=c+2\pi\int_0^x H\psi(y)\,dy$.)

Formula (2.54) shows in particular that

$$\operatorname{supp}(\psi) = \{x : q^{(0)}(x) < 0\}. \tag{3.12}$$

The proof of this fact which we presented above (that is, the proof of Proposition 2.51) follows from the analyticity properties of the Cauchy transform $F(z)$ of $\psi(x)\,dx$, or, in the case that V is real analytic in a neighborhood of $[-1, 1]$, from the Ricatti equation associated with (2.4) in the standard way. It is of interest to give a direct proof of (3.12) using ode methods. We first prove the following Lemma concerning the potential (2.14).

LEMMA 3.13. *For* $x \in (-1, 1)$, *the potential defined in* (2.14) *satisfies*

$$\lim_{n\to\infty} \frac{q^{(n)}(x)}{n^2} = q^{(0)}(x) = \left(\frac{V'(x)}{2}\right)^2 + \frac{1}{x^2-1}\left[1 + \int_{-1}^{1}\int_0^1 h(tx + (1-t)\,y)\,dt\,\psi(y)\,dy\right], \tag{3.14}$$

where

$$h(u) = \frac{d}{du}\left(V'(u)(u^2-1)\right). \tag{3.15}$$

A simple calculation shows that $q^{(0)}(x)$ is precisely the function defined in (1.37).

Proof. From the definition (2.14) of $q^{(n)}(x)$, it is clear that we only need to prove that

$$\frac{Q(x)}{n^2} \to 1 + \mu_{ES}^V\left(\int_0^1 h(tx + (1-t)\cdot)\,dt\right). \tag{3.16}$$

But from the second line of equation (2.5), we have

$$\frac{Q}{n^2} = 1 - \frac{1}{n} + \frac{n-1}{n}\,\rho_{S^{(n)}}(H(x, \cdot)), \tag{3.17}$$

where

$$H(x, y) = \int_0^1 h(tx + (1-t)\,y)\,dt \tag{3.18}$$

and, as $H(x, y)$ is a continuous function of y, the Lemma is proved. ∎

But more is true.

LEMMA 3.19. *The potential* $(1/n^2)\, q^{(n)}(x)$ *converges uniformly to* $q^{(0)}(x)$ *on compact subsets of* $(-1, 1)$.

Proof. Again, the only difficulty lies in proving the uniform convergence of the quantity $Q(x)/n^2$. Moreover, from (3.17), we see that we need only establish the uniform convergence of $\rho_{S^{(n)}}(H(x, \cdot))$. We proceed in two steps.

First, we will assume that $h(u)$ in (3.15) is a polynomial of degree M. But this implies that $H(x, y)$ defined in (3.18) is a polynomial in x of degree M, with coefficients which are polynomials in y. More precisely, we have

$$H(x, y) = \sum_{i=0}^{M} x^i c_i(y), \tag{3.20}$$

$$\text{degree}(c_i(y)) = M - i. \tag{3.21}$$

Thus, we have,

$$\rho_{S^{(n)}}(H(x, \cdot)) = \sum_{i=0}^{M} x^i \rho_n(c_i(\cdot)), \tag{322}$$

and the uniform convergence is now immediate.

We now complete the proof of the Lemma. Given $\varepsilon > 0$, we approximate $h(u)$ by a polynomial $h_{app}(u)$ such that

$$\|h - h_{app}\|_\infty = \sup_{|u| \leqslant 1} |h(u) - h_{app}(u)| < \frac{\varepsilon}{3}, \tag{3.23}$$

which implies that

$$\sup_{|x|,\, |y| \leqslant 1} |H(x, y) - H_{app}(x, y)| < \frac{\varepsilon}{3}, \tag{3.24}$$

where

$$H_{app}(x, y) = \int_0^1 h_{app}(tx + (1-t)\, y)\, dt. \tag{3.25}$$

But then

$$\|\rho_n(H(x, \cdot)) - \rho_n(H_{app}(x, \cdot))\|_\infty \leqslant \frac{\varepsilon}{3} \tag{3.26}$$

for *all* $n \in \mathbf{N}$, and hence

$$\|\mu_{ES}^V(H(x,\cdot)) - \mu_{ES}^V(H_{app}(x,\cdot))\|_\infty \leqslant \frac{\varepsilon}{3}. \tag{3.27}$$

Next, from the uniform convergence of $\rho_n(H_{app}(x,\cdot))$, we take N such that for all $n > N$, we have

$$\|\rho_n(H_{app}(x,\cdot)) - \mu_{ES}^V(H_{app}(x,\cdot))\|_\infty \leqslant \frac{\varepsilon}{3}. \tag{3.28}$$

A standard $\varepsilon/3$ argument completes the proof. ▮

Summary. We have now established that the function g defined in (2.12) satisfies the "semi-classical" differential equation

$$-\frac{1}{n^2} g'' + \left[q^{(0)}(x) + \frac{\alpha_n(x)}{x^2 - 1} \right] g(x) = 0, \tag{3.29}$$

where now $\alpha_n(x) \to 0$ uniformly for $x \in [-1, 1]$.

Intuitively, one expects that the function g can have roots *only* where the potential $q^{(0)}(x)$ is negative, i.e. one expects the roots of g to accumulate on those subintervals of the interval $[-1, 1]$ where $q^{(0)}(x) < 0$.

LEMMA 3.30. *Let* $[a, b] \subset (-1, 1)$ *be an interval on which*

$$q^{(0)}(x) < \gamma < 0. \tag{3.31}$$

Then

$$\psi(x) > 0 \qquad \text{for a.e.} \quad x \in (a, b). \tag{3.32}$$

Proof. We first take n so large that

$$q^{(n)}(x) = q^{(0)}(x) + \frac{\alpha_n(x)}{x^2 - 1} < \frac{\gamma}{2} \tag{3.33}$$

for $x \in [a, b]$.

Let y solve

$$-\frac{1}{n^2} y'' + \frac{\gamma}{2} y = 0. \tag{3.34}$$

It then follows from the Sturm oscillation theorem [4, Chapter 8] that g must vanish in between consecutive roots of any solution of (3.34), as $q < \gamma/2$. Taking the particular solution

$$y = \sin\left(n(x-a)\sqrt{-\frac{\gamma}{2}}\right), \tag{3.35}$$

we see that

$$\int_a^b \psi\, dx \geqslant \frac{1}{\pi}(b-a)\sqrt{\frac{-\gamma}{2}}. \tag{3.36}$$

Choosing now an arbitrary subinterval $(a', b') \subset (a, b)$, we deduce that

$$\int_{a'}^{b'} \psi\, dx \geqslant \frac{1}{\pi}(b'-a')\sqrt{\frac{-\gamma}{2}}, \tag{3.37}$$

and hence since the subinterval (a', b') was arbitrary, we must have

$$\psi(x) > 0 \qquad \text{for a.e.} \quad x \in \{x\colon q^{(0)}(x) < 0\}. \quad \blacksquare \tag{3.38}$$

COROLLARY 3.39. *We have*

$$\psi > 0 \qquad \textit{for a.e.} \quad x \in \{x\colon q^{(0)}(x) < 0\}. \tag{3.40}$$

LEMMA 3.41. *Let* $[a, b] \subset (-1, 1)$ *be any interval on which*

$$q^{(0)}(x) \geqslant \hat{\gamma} > 0. \tag{3.42}$$

Then

$$\psi(x) = 0 \qquad \textit{for a.e.} \quad x \in (a, b). \tag{3.43}$$

Proof. We take n so large that

$$q(x) = q^{(0)}(x) + \frac{\alpha_n(x)}{x^2-1} > \frac{\hat{\gamma}}{2} \tag{3.44}$$

for $x \in [a, b]$. As in the proof of the previous Lemma, let y solve

$$-\frac{1}{n^2}y'' + \frac{\hat{\gamma}}{2}y = 0. \tag{3.45}$$

It then follows from a comparison argument (cf. Lemma 3.30, [4, Chapter 8]) that if g vanishes twice in (a, b), then any solution of (3.45) must vanish between the roots of g. Take

$$y(x) = e^n \sqrt{(\hat{\gamma}/2)}\, x. \tag{3.46}$$

Then, as y does not vanish, g can have at most 1 root in (a, b). Thus,

$$\int_a^b \psi(x)\, dx = 0. \tag{3.47}$$

As $\psi \geqslant 0$, the Lemma is proved. ∎

COROLLARY 3.48. *We have*

$$\psi(x) = 0 \qquad \textit{for a.e.} \quad x \in \{x: q^{(0)}(x) > 0\}. \tag{3.49}$$

Remark. The Corollaries to the two preceding Lemmas constitute an ode proof of (3.12).

4. PROOF OF THEOREM 1.46 ON THE NUMBER OF INTERVALS IN THE SUPPORT OF THE EQUILIBRIUM MEASURE, AND OTHER APPLICATIONS OF PROPOSITION 2.51

If V is a polynomial of degree m, then

$$q^{(0)}(x) = \frac{r(x)}{x^2 - 1}, \tag{4.1}$$

where $r(x)$ is a polynomial of degree $2m$. Indeed,

$$q^{(0)}(x) = \left(\frac{V'(x)}{2}\right)^2 + \frac{1}{x^2 - 1}\,[1 + (\mu^V_{ES}(H(x, \cdot)))], \tag{4.2}$$

and $H(x, y)$ is a polynomial of degree m (c.f. (3.18) and (3.15)). Now we leave the reader to verify that the maximum number of intervals in $[-1, 1]$ where $q^{(0)}$ could be negative is $m + 1$. This proves Theorem 1.46.

The reader will observe from the proof of formula (1.37) (see, in particular, Lemma 2.3) that if the n-Fekete points $x_1^*, ..., x_n^*$ all lie within $(-1, 1)$ as $n \to \infty$, then there is no need to introduce the factor $x^2 - 1$ into the definition of Q, and one again obtains formula (1.37) for $q^{(0)}(x)$, but now without the polar term $(1/(x^2 - 1))(1 + \int V'(y)(x + y)\, \psi(y)\, dy)$. This means that in the cases where the support of $\psi(x)\, dx$ lies away from the boundary in a compact subset of $(-1, 1)$, the term $1 + \int V'(y)(x+y)\,\psi(y)\,dy$ must necessarily vanish. We see this as follows.

First note from Lemma 2.45 that

$$V'(y)\,\psi(y) = -2\pi(H\psi(y))\,\psi(y) \tag{4.3}$$

$$= \frac{-\pi}{2i}\,(F_+^2(y) - F_-^2(y)) \tag{4.4.}$$

as $F_\pm = \pm\psi + iH\psi$. But $F_\pm^2(y) \in L^1(a, b)$ for any subinterval $[a, b] \subset (-1, 1)$ (cf. proof of Proposition 2.51).

Thus

$$\int V'(y)(x+y)\,\psi(y)\,dy \tag{4.5}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{-\pi}{2i} \int_{-1+\varepsilon}^{1-\varepsilon} (x+y)(F_+^2(y) - F_-^2(y))\,dy \tag{4.6}$$

$$= \frac{\pi}{2i} \oint (x+z)\,F^2(z)\,dz - \frac{\pi}{2i} \lim_{\varepsilon \downarrow 0} \oint_{|z-1|=\varepsilon} (x+z)\,F^2(z)\,dz \tag{4.7}$$

$$- \frac{\pi}{2i} \lim_{\varepsilon \downarrow 0} \oint_{|z+1|=\varepsilon} (x+z)\,F^2(z)\,dz. \tag{4.8}$$

Here all three contours are counterclockwise and the first contour encloses $[-1, 1]$. Now $F(z) \sim 1/(-i\pi z)$ as $z \to \infty$, and by contour integration we see that the first integral is just equal to -1. On the other hand the calculations in Proposition 2.51 imply, in particular, that

$$\lim_{\varepsilon \downarrow 0} \oint_{|z-1|=\varepsilon} (z-1)\,F^2(z)\,dz = 0 = \lim_{\varepsilon \downarrow 0} \oint_{|z+1|=\varepsilon} (z+1)\,F^2(z)\,dz.$$

Thus

$$\frac{1}{x^2-1}\left(1 + \int V'(y)(x+y)\,\psi(y)\,dy\right) \tag{4.9}$$

$$= \frac{-\pi}{2i(x-1)} \lim_{\varepsilon \downarrow 0} \oint_{|z-1|=\varepsilon} F^2(z)\,dz \tag{4.10}$$

$$+ \frac{-\pi}{2i(x+1)} \lim_{\varepsilon \downarrow 0} \oint_{|z+1|=\varepsilon} F^2(z)\,dz, \tag{4.11}$$

and hence

$$q^{(0)}(x) = \left(\frac{V'(x)}{2}\right)^2 + \int \frac{V'(x) - V'(y)}{x - y}\,\psi(y)\,dy + \frac{c_+}{x-1} + \frac{c_-}{x+1} \tag{4.12}$$

$$= \left(\frac{V'(x)}{2}\right)^2 + \int_{-1}^{1} \left(\int_0^1 V''(sx + (1-s)\,y)\,ds\right) \psi(y)\,dy$$

$$+ \frac{c_+}{x-1} + \frac{c_-}{x+1}, \tag{4.13}$$

where

$$c_+ = \frac{-\pi}{2i} \lim_{\varepsilon \downarrow 0} \oint_{|z-1| = \varepsilon} F^2(z)\,dz, \qquad c_- = \frac{-\pi}{2i} \lim_{\varepsilon \downarrow 0} \oint_{|z+1| = \varepsilon} F^2(z)\,dz. \tag{4.14}$$

Necessarily $c_+ \geqslant 0$ and $c_- \leqslant 0$. Indeed if $c_+ < 0$ (the case $c_- > 0$ is similar), then $q^{(0)}(x) > 0$ for x near 1 and so the support of $\psi(x)\,dx$ lies in a compact subinterval of $[-1, 1]$. But then from the relation $F(z) = (1/\pi i) \int (\psi(y)\,dy)/y - z$, we see that $F(z)$ is analytic in a neighborhood of $z = 1$, which implies in turn that $c_+ = 0$, by (4.14). Hence

$$c_+ \geqslant 0 \qquad \text{and} \qquad c\_0. \tag{4.15}$$

In particular this means that the terms $c_+/(x-1)$ and $c_-/(x+1)$ always make a *negative* contribution to $q^{(0)}(x)$.

The preceding argument clearly proves, in addition, the following result, which resolves the question concerning the polar term raised above.

PROPOSITION 4.16. *If the support of μ_{ES}^V lies in a compact subinterval of $[-1, 1)$ (respectively $(-1, 1]$) then $c_+ = 0$, (respectively $c_- = 0$). In particular if the support of μ_{ES}^V lies in a compact subinterval of $(-1, 1)$, then*

$$q^{(0)}(x) = \left(\frac{V'(x)}{2}\right)^2 + \int \frac{V'(x) - V'(y)}{x - y}\,\psi(y)\,dy$$

$$= \left(\frac{V'(x)}{2}\right)^2 + \int_{-1}^{1} \left(\int_0^1 V''(sx + (1-s)\,y)\,ds\right) \psi(y)\,dy. \tag{4.17}$$

There are a number of immediate consequences of the above results.

PROPOSITION 4.18. (i) *If $V''(x) \leqslant 0$ for all $x \in [-1, 1]$, and $\hat{x} \in [-1, 1]$ is a critical point of V, i.e., $V'(\hat{x}) = 0$, then $\hat{x} \in \operatorname{supp} \mu_{ES}^V$.*

(ii) *If* $V''(x) \leqslant \frac{1}{2}$ *for all* $x \in [-1, 1]$, *then the closure of supp* μ_{ES}^{V} *is precisely one interval.*

(iii) *If* $V''(x) \geqslant -1/4$ *for all* $x \in [-1, 1]$, *then supp* μ_{ESS}^{V} *is not compactly contained in* $(-1, 1)$.

(iv) *If V is even, and* $V'(x) \geqslant 0$ *for all* $0 < x \leqslant 1$, *then supp*(μ_{ES}^{V}) *is not compactly contained in* $(-1, 1)$.

Remarks. If $V''(x) \leqslant 0$ and V is even then $0 \in supp\, \mu_{ES}^{V}$ by (i) above. This implies that for such potentials fast decreasing polynomials exist: however, this is a trivial implication, as $V(x) = V(-x)$, $V''(x) \leqslant 0$ implies that $V(x) \leqslant 0$, as $V(0) = 0$ for the problem of fast decreasing polynomials (Problem 5).

Note also from (ii) and (iii) that if $-1/4 \leqslant V''(x) \leqslant \frac{1}{2}$, then the support of μ_{ES}^{V} is either $(-1, \beta_0)$, $(\alpha_0, 1)$, or $(-1, 1)$. In addition if V is even then the support must be $(-1, 1)$. Finally we note that an additional general application of (1.36) (i.e. Proposition 2.51) is given in Lemma 6.2 below.

Additional Remark. Parts (i) and (iv) above are special cases of the well-known fact that if the maximum of V is attained at $\hat{x}$, then $\hat{x} \in \mathrm{supp}(\mu_{ES}^{V})$ (see [28]).

Proof. Suppose first that $\hat{x} \in (-1, 1)$. As $V'(\hat{x}) = 0$, we see from (4.13) and (4.15) that $q^{(0)}(\hat{x}) \leqslant 0$. If $q^{(0)}(\hat{x}) = 0$, then $c_+ = c_- = 0$ and $\int (V'(\hat{x}) - V'(y))/(\hat{x} - y)\, \psi(y)\, dy = 0$. But then $(V'(\hat{x}) - V'(y))\, \psi(y) = -V'(y)\, \psi(y) = 0$ a.e. as $V'' \leqslant 0$ implies $(V'(\hat{x}) - V'(y))/(\hat{x} - y) \leqslant 0$. But then by (2.46), $\psi(y)(H\psi(y)) = (-1/2\pi)\, \psi(y)\, V'(y) = 0$ a.e., and hence $F_+^2(y) - F_-^2(y) = 0$ a.e. Thus $F^2(z)$ is analytic across $(-1, 1)$: moreover as $c_+ = c_- = 0$, and $F^2(z)$ has at worst simple poles at ± 1, it follows from (4.14) that in fact $F^2(z)$ is entire. But $F(z) \to 0$ as $z \to \infty$ and hence $F(z) \equiv 0$, which contradicts $\int \psi = 1$. Thus $q^{(0)}(\hat{x}) < 0$.

If $\hat{x} = 1$, and $c_+ > 0$, then we are already done. If $c_+ = 0$, then $q^{(0)}(x)$ is continuous up to 1 and $q^{(0)}(1) \leqslant 0$. If $q^{(0)}(1) = 0$, we argue as above, etc.

We now turn to case (ii). Denote by $\hat{J}$ the closure of $supp\, \mu_{ES}^{V}$. Suppose there exists $x \notin \hat{J}$, and $y_1, y_2 \in \hat{J} \cap (-1, 1)$, with $y_1 < x < y_2$. We will show this leads to a contradiction. Firstly, there must be an open interval $B(x) \ni x$, $B(x) \cap \hat{J}$ empty. Denote by $\hat{B}(x) = (y_*, y^*)$ the union of all such intervals. Clearly $y_1 \leqslant y_* < y^* \leqslant y_2$. We see from the definition $H\psi(y) = (1/\pi) \int_{\hat{J}} (\psi(t)/(y - t))\, dt$ that $H\psi$ is differentiable at y, $y \in \hat{B}(x)$ and moreover

$$\frac{d}{dy} H\psi(y) = \frac{-1}{\pi} \int_{\hat{J}} \frac{\psi(t)}{(y - t)^2}\, dt < -\frac{1}{4\pi} \qquad \textit{for all} \quad y \in \hat{B}(x), \tag{4.19}$$

as $|y-t|<2$, and $\int \psi = 1$. Thus we see that

$$\frac{d}{dy}\left(H\psi(y)+\frac{V'(y)}{2\pi}\right)<0 \qquad \text{for} \quad y \in \hat{B}(x). \tag{4.20}$$

Secondly, as $y_*, y^* \in \hat{J}$, the continuity of $q^{(0)}$ implies that $q^{(0)}(y_*)=0=q^{(0)}(y^*)$. Thus we conclude from (2.53) that

$$H\psi(y_*)+\frac{V'(y_*)}{2\pi}=0=H\psi(y^*)+\frac{V'(y^*)}{2\pi},$$

which contradicts (4.20).

Therefore if $x \notin \hat{J}$, then x must lie to the left or to the right of $\hat{J}$, i.e., $\hat{J}$ must be an interval.

We now prove case (iii). If $c_+>0$ and/or $c_-<0$ then the support of $d\mu_{ES}^V$ clearly extends to the boundary. If $c_+=c_-=0$, then from (4.13) we see that $q^{(0)}(x) \geqslant -1/4$ for all x, and hence $\psi(x) \leqslant \frac{1}{2}$ a.e. But then $1=\int \psi \leqslant 1$ which is possible only if $\psi(x)=\frac{1}{2}$ a.e. While this result of course also implies that the support of $d\mu_{ES}^V$ extends to the boundary, it in fact cannot occur. Indeed $\psi(x)=\frac{1}{2}$ a.e. implies $V'(x)=2\pi H\psi(x)=\log(1+x)/(1-x)$, which contradicts $V \in C^2$. Thus we see that the support of $d\mu_{ES}^V$ extends to the boundary, and $\psi(x)$ has a square root singularity at $x=1$ or $x=-1$ (or both).

We finish the proof with case (iv). For any $V \in C^2$, we have

$$q^{(0)}(x) \leqslant c_1+\frac{1}{x^2-1}\left(1+\int V'(y)(x+y)\, d\mu_{ES}^V(y)\right).$$

Moreover, as $V(x)$ is even, $d\mu_{ES}^V(y)$ is necessarily even by the uniqueness of the equilibrium measure, and hence $\int V'(y)\, d\mu_{ES}^V(y)=0$. As $V'(y)\, y \geqslant 0$ for all y, it follows that $q^{(0)}(x) \leqslant c_1+1/(x^2-1)$. Hence $q^{(0)}(x)<0$ for $(x^{\#})^2<x^2<1$ for some constant $x^{\#}$, $0<x^{\#}<1$, which depends only on V. This proves (iv). ∎

5. VARIATIONAL EQUATIONS

In this short Section we will summarize, for convenience, the variational conditions satisfied by the equilibrium measure μ_{ES}^V. The statements contained in this Section, with their proofs, can be found, for example, in [19] or [28]. These statements were mentioned briefly in the Introduction (see the discussion of Problem 1), and used in part in Lemma 2.45. Rather than considering the equilibrium measure μ_{ES}^V directly, it is convenient to phrase the variational conditions in terms of the maximizer ψ, $d\mu_{ES}^V=\psi(x)\, dx$.

As usual, we denote the support of ψ by J. In the case that V is real analytic in a neighborhood of $[-1, 1]$, we have $J = \bigcup_{j=0}^{N} (\alpha_j, \beta_j)$, where $-1 \leqslant \alpha_0 < \beta_0 < \alpha_1 \cdots < \alpha_N < \beta_N \leqslant 1$.

LEMMA 5.1. *A function $\psi \geqslant 0$ in $L_{1/2,\infty}$ is the maximizer if and only if there exists $\ell \in \mathbf{R}$ such that ψ satisfies*

$$L\psi(x) + V(x) = \int \log(x-y)^2 \psi(y)\, dy + V(x) = \ell \qquad \textit{for} \quad x \in J, \tag{5.2}$$

$$\psi(x) > 0 \qquad \textit{for a.e.} \quad x \in J, \quad \int \psi(x)\, dx = 1, \tag{5.3}$$

$$L\psi(x) + V(x) \leqslant \ell \qquad \textit{for all} \quad x \in [-1, 1], \tag{5.4}$$

and

$$\psi(x) = 0 \qquad \textit{for a.e.} \quad x \in \mathbf{R} \backslash J. \tag{5.5}$$

Note that, by Lemma 2.45, we can differentiate equation (5.2), and we obtain

$$H\psi + \frac{V'(x)}{2\pi} = 0 \qquad \textit{for a.e.} \quad x \in J. \tag{5.6}$$

Furthermore, in the case that J is a finite union of intervals, inequalities (5.3) *and* (5.4) *can be expressed as*

$$\psi(x) > 0 \qquad \textit{for a.e.} \quad x \in J, \tag{5.7}$$

$$\int_{\beta_j}^{y} \left(H\psi(x) + \frac{V'(x)}{2\pi} \right) dx \leqslant 0 \qquad \textit{for} \quad y \in (\beta_j, \alpha_{j+1}), \tag{5.8}$$

with equality for $y = \alpha_{j+1}$, $j < N$,

$$\int_{y}^{\alpha_0} \left(H\psi(x) + \frac{V'(x)}{2\pi} \right) dx \geqslant 0 \qquad \textit{for} \quad y \in (-1, \alpha_0), \tag{5.9}$$

$$\psi(x) = 0 \qquad \textit{on } \mathbf{R} \backslash J, \tag{5.10}$$

where we take $\alpha_{N+1} = 1$.

Thus we have the following corollary of Lemma 5.1.

LEMMA 5.11. *If V is real analytic in a neighborhood of $[-1, 1]$ the maximizer ψ is uniquely determined by the conditions* (5.6–5.10).

In what follows we will at times present a candidate ψ, and then verify directly that ψ solves (5.6)–(5.10), in order to prove that the function ψ considered is indeed the maximizer. At other times, we will use the potential $q^{(0)}$ to identify the support, and then produce ψ.

6. PROOF OF THEOREM 1.48—THE CASE $V(x) = -tx^{2q}$, $t>0$

We begin this section by supposing that the external field $V(x)$ is a monomial:

$$V(x) = tx^m. \tag{6.1}$$

Our first goal is to prove that for t small, the support of ψ is the entire interval $(-1, 1)$.

LEMMA 6.2. *Let* $V(x) = tx^m$, *m a positive integer,* $t \in \mathbf{R}$. *Then there is an* $\varepsilon > 0$ *such that for all* $|t| < \varepsilon$, *the support J of the maximizer ψ is the interval* $(-1, 1)$.

Proof. In this case, the potential $q^{(0)}$ is (cf. Lemma 3.13)

$$q^{(0)} = \left(\frac{mt}{2}\right)^2 x^{2m-2} + mt \int \psi(y)(x^{m-2} + x^{m-3}y + \cdots + y^{m-2})\, dy$$
$$+ \frac{1}{x^2-1}\left[1 + mt \int_{-1}^{1} \psi(y)(xy^{m-1} + y^m)\, dy\right]. \tag{6.3}$$

Clearly, for t sufficiently small, we have

$$q^{(0)}(x) < 0 \qquad \text{for all} \quad x \in (-1, 1). \quad \blacksquare \tag{6.4}$$

Remark. This Lemma implies that for t small, the maximizer ψ is described by case (i) of Theorem 1.38, for *all* monomial fields. In fact, it is clear from the above that the maximizer ψ is described by case (i) of Theorem 1.34 for *all external fields that are sufficiently small.*

We now turn to Theorem 1.48. Thus we take $m = 2q$ (q an integer), and consider

$$V(x) = -tx^{2q}, \qquad t > 0. \tag{6.5}$$

From Lemma 6.2 and Theorem 1.38, we know that for $0 < t \ll 1$, the maximizer given by

$$\psi(x) = ReF_+(x), \tag{6.6}$$

where

$$F(z) = (z^2-1)^{1/2} \frac{1}{\pi i} \int_{-1}^{1} \frac{qti}{\pi} \frac{y^{2q-1}}{(y^2-1)^{1/2}} \frac{dy}{y-z} + \frac{i\gamma}{\pi} (z^2-1)^{-1/2}, \qquad (6.7)$$

with γ chosen so that $\int_{-1}^{1} \psi(x)\, dx = 1$ (note that formula (1.39) rather than (1.40) is appropriate as $\int_{-1}^{1} (y^{2q-1}/(y^2-1)_+^{1/2})\, dy = 0$). The integral in (6.7) can be computed by a residue calculation, and we obtain

$$F(z) = \frac{qti}{\pi} [z^{2q-1} - (z^2-1)^{1/2} h_0(z)] + \frac{i\gamma}{\pi} (z^2-1)^{-1/2}, \qquad (6.8)$$

where $h_0(z)$ is given by

$$\begin{aligned} h_0(z) &= \frac{1}{2\pi i} \oint \frac{y^{2q-1}}{(y^2-1)^{1/2}} \frac{dy}{y-z} \\ &= z^{2q-2} + \sum_{j=1}^{q-1} z^{2q-2-2j} \prod_{\ell=1}^{j} \frac{2\ell-1}{2\ell}. \end{aligned} \qquad (6.9)$$

Here the integral is taken over a closed counterclockwise loop which contains the interval $[-1, 1]$, as well as the point z.

From (6.8), we have the following expression for ψ, for $x \in (-1, 1)$:

$$\psi(x) = -\frac{qti}{\pi} (x^2-1)^{1/2} h_0(x) + \frac{i\gamma}{\pi} (x^2-1)^{-1/2}. \qquad (6.10)$$

We note from (6.10) and (6.9) that

$$\int_{-1}^{1} \psi(x)\, dx = qt \prod_{\ell=1}^{q} \frac{2\ell-1}{2\ell} + \gamma, \qquad (6.11)$$

and hence for t small, the condition $\int_{-1}^{1} \psi(x)\, dx = 1$ implies that $\gamma > 0$. Also we see from (6.9) and (6.10) that for t small

$$\psi(x) > 0 \qquad \text{for all} \quad x \in (-1, 1). \qquad (6.12)$$

Remark. We see from the above that for $0 < t \ll 1$, ψ satisfies the conditions (5.6)–(5.10), and hence ψ is the maximizer, by Lemma 5.11. In other words, the above calculations provide a direct and independent verification of Lemma 6.2 in the case $V(x) = -tx^{2q}$, $t > 0$.

As we increase t, we see that the function ψ defined by (6.10) continues to satisfy conditions (5.6)–(5.10) for $0 < t < t_{-,2q}$, where $t_{-,2q}$ is defined to be the value of t where $\gamma = 0$, i.e.,

$$\gamma = 1 - qt_{-,2q} \prod_{\ell=1}^{q} \frac{2\ell - 1}{2\ell} = 0, \tag{6.13}$$

or $t_{-,2q} = (q \prod_{\ell=1}^{q} (2\ell - 1)/(2\ell))^{-1}$. However, for $t > t_{-,2q}$, $\gamma < 0$, and it is clear from (6.10) that $\psi(x)$ will be negative near ± 1. Thus, for $t > t_{-,2q}$, ψ does not satisfy condition (5.7).

For $t > t_{-,2q}$, we will suppose that $J = (-\beta_0, \beta_0)$, with $-\alpha_0 = \beta_0 = \beta < 1$. We will prove that this is indeed the support of ψ by establishing that the corresponding ψ, defined using Theorem 1.38, satisfies conditions (5.6)–(5.10), and hence is the maximizer.

Remark. Because the external field V is symmetric, and $V''(x) \leqslant 0$, we know from Proposition 4.18 that the support J must be a single interval $(-\beta, \beta)$, $\beta \leqslant 1$, for all $t > 0$. Here we present an independent verification of this fact, by directly verifying that the corresponding ψ solves the maximization problem.

From Theorem 1.38, we see that we are in case (v). Thus we have

$$\psi(x) = F_+(x), \tag{6.14}$$

where now $F(z)$ is defined by

$$F(z) = (z^2 - \beta^2)^{1/2} \frac{1}{\pi i} \int_{-\beta}^{\beta} \frac{qti}{\pi} \frac{y^{2q-1}}{(y^2 - \beta^2)^{1/2}} \frac{dy}{y - z}, \tag{6.15}$$

and β must be chosen to satisfy $\int_{-1}^{1} \psi \, dx = 1$. The integral in (6.15) can be computed by a residue calculation, and we obtain

$$F(z) = \frac{qti}{\pi} [z^{2q-1} - (z^2 - \beta^2)^{1/2} h_1(z)], \tag{6.16}$$

where $h_1(z)$ is given by

$$h_1(z) = \frac{1}{2\pi i} \oint \frac{y^{2q-1}}{(y^2 - \beta^2)^{1/2}} \frac{dy}{y - z} \tag{6.17}$$

$$= z^{2q-2} + \sum_{j=1}^{q-1} z^{2q-2-2j} \beta^{2j} \prod_{\ell=1}^{j} \frac{2\ell - 1}{2\ell}. \tag{6.18}$$

The integral in (6.17) is taken over a closed loop which contains the interval $[-\beta, \beta]$, as well as the point z. From (6.16), we have the following expression for $\psi(x)$, $x \in (-\beta, \beta)$:

$$\psi(x) = -\frac{qti}{\pi}(x^2 - \beta^2)^{1/2}\, h_1(x). \tag{6.19}$$

A simple contour integration shows that

$$\int_{-1}^{1} \psi(x)\, dx = \int_{-\beta}^{\beta} \psi(x)\, dx = qt\beta^{2q} \prod_{l=1}^{q} \frac{2\ell - 1}{2\ell}, \tag{6.20}$$

and hence $\int_{-1}^{1} \psi(x)\, dx = 1$ implies that

$$\beta = \left(qt \prod_{\ell=1}^{q} \frac{2\ell - 1}{2\ell} \right)^{-1/2q} = \left(\frac{t_{-,2q}}{t} \right)^{1/2q}. \tag{6.21}$$

It is clear from (6.21) that $\beta < 1$ for $t > t_{-,2q}$. Moreover, from (6.18), we see that $h_1(x) > 0$ for all x. Hence

$$\psi(x) > 0 \qquad \text{for} \quad x \in (-\beta, \beta), \tag{6.22}$$

and furthermore, using the relation $F_+(x) = \psi(x) + i(H\psi)(x)$, we obtain from (6.16), for $y \in (\beta, 1)$,

$$\int_{\beta}^{y} \left(H\psi(x) - \frac{tq}{\pi} x^{2q-1} \right) dx = \int_{\beta}^{y} -\frac{tq}{\pi}(x^2 - \beta^2)^{1/2}\, h_1(x)\, dx \leqslant 0. \tag{6.23}$$

Similarly, we have

$$\int_{y}^{-\beta} H\psi(x) - \frac{tq}{\pi} x^{2q-1}\, dx \geqslant 0 \tag{6.24}$$

for all $y \in (-1, -\beta)$. This proves that $\psi(x)$ defined by (6.14) satisfies conditions (5.6)–(5.10), and hence is the maximizer. Thus $J = (-\beta, \beta)$ for *all* $t > t_{-,2q}$. Moreover, we see from (6.21) that for $t > t_{-,2q}$, $\beta(t)$ is analytic in t, and as $t \to \infty$, $\beta \to 0$. Also as $\lim_{t \downarrow t_{-,2q}} \beta(t) = 1$, we see that $\beta(t)$ is continuous for all $t > 0$.

This completes the proof of Theorem 1.48.

7. PROOF OF THEOREM 1.52—THE CASE $V(x)=tx^{2q}$, $t>0$.

We take $m=2q$, and $V=tx^{2q}$, $t>0$. From Lemma 6.2, we know that for $0<t\ll 1$, $\sigma_{ES}^{V}=[-1,1]$, and hence (cf. (6.6)–(6.7)),

$$\psi(x)=ReF_{+}(x), \tag{7.1}$$

where

$$F(z)=(z^2-1)^{1/2}\frac{1}{\pi i}\int_{-1}^{1}\frac{-qti}{\pi}\frac{y^{2q-1}}{(y^2-1)^{1/2}}\frac{dy}{y-z}+\frac{i\gamma}{\pi}(z^2-1)^{-1/2}, \tag{7.2}$$

and again γ is chosen such that $\int_{-1}^{1}\psi(x)\,dx=1$. As in (6.8),

$$F(z)=-\frac{qti}{\pi}[z^{2q-1}-(z^2-1)^{1/2}h_0(z)]+\frac{i\gamma}{\pi}(z^2-1)^{-1/2}, \tag{7.3}$$

where $h_0(z)$ is again given by (6.9). Thus, for $x\in(-1,1)$, we have

$$\psi(x)=\frac{i\gamma}{\pi}(x^2-1)^{-1/2}+\frac{qti}{\pi}(x^2-1)^{1/2}h_0(x). \tag{7.4}$$

The condition $\int\psi=1$ requires

$$\gamma-qt\prod_{\ell=1}^{q}\frac{2\ell-1}{2\ell}=1, \tag{7.5}$$

which implies that $\gamma>1$ for all $t>0$. Moreover, it is clear from (7.4) that for t small $\psi(x)>0$ for all $x\in(-1,1)$, and hence ψ satisfies all of conditions (5.6)–(5.10). Once again this is a direct verification of Lemma 6.2 for $0<t\ll 1$.

As we increase t, $\psi(x)$ as given by (7.4) becomes negative for some $x\in(-1,1)$. Indeed, observe from (6.9), (7.4), and (7.5), that

$$\begin{aligned}\psi(0)&=\frac{\gamma}{\pi}-\frac{qt}{\pi}\prod_{\ell=1}^{q-1}\frac{2\ell-1}{2\ell}\\ &=\frac{1}{\pi}\left(1+qt\left(\prod_{\ell=1}^{q}\frac{2\ell-1}{2\ell}-\prod_{\ell=1}^{q-1}\frac{2\ell-1}{2\ell}\right)\right)\\ &=\frac{1}{\pi}\left(1-\frac{t}{2}\prod_{\ell=1}^{q-1}\frac{2\ell-1}{2\ell}\right),\end{aligned} \tag{7.6}$$

and hence for t sufficiently large, $\psi(0)$ will be negative. This implies that there must exist a critical value $t_{+,2q}$ such that for all $0<t<t_{+,2q}$, ψ defined by (7.4) is the maximizer, while for $t_{+,2q}<t$, ψ defined by (7.4) is not the maximizer. This is made precise in the following 3 Lemmas.

First, note that substituting (7.5) into (7.4) we have the following useful representation for ψ,

$$\psi=\frac{i}{\pi(z^2-1)^{1/2}}\,[1+qtr_0(z)], \tag{7.7}$$

where

$$r_0(z)=\left((z^2-1)\,h_0(z)+\prod_{\ell=1}^{q}\frac{2\ell-1}{2\ell}\right). \tag{7.8}$$

Using the polynomial expression for $h_0(z)$ in (6.9), we obtain

$$r_0(z)=z^{2q}-\left(\frac{z^{2q-2}}{2}+\sum_{j=1}^{q-1}\frac{z^{2q-2-2j}}{2(j+1)}\prod_{\ell=1}^{j}\frac{2\ell-1}{2\ell}\right) \tag{7.9}$$

and using the integral expression for $h_0(z)$, together with the symmetry $h_0(z)=h_0(-z)$, we obtain

$$r_0(z)=\frac{1}{2\pi i}\oint y^{2q}(y^2-1)^{1/2}\,\frac{dy}{y^2-z^2}, \tag{7.10}$$

where the contour integral in (7.10) is taken in the counter-clockwise direction on a circle of large radius.

LEMMA 7.11. *Let*

$$h(x)=x^m-\sum_{j=0}^{m-1}a_jx^j \tag{7.12}$$

with $a_j\geqslant 0$, and one of $\{a_j\}$ strictly positive. Suppose that $x>0$ and $h'(x)=0$. Then $h''(x)>0$.

Proof. First, if $a_j=0$ for $j=1, ..., m-1$, then the result is trivially true. Otherwise, we have

$$h'(x)=mx^{m-1}-\sum_{j=0}^{m-1}ja_jx^{j-1}, \tag{7.13}$$

$$h''(x)=m(m-1)\,x^{m-2}-\sum_{j=0}^{m-1}j(j-1)\,a_jx^{j-2}, \tag{7.14}$$

and hence

$$xh''(x) = (m-1) \sum_{j=0}^{m-1} ja_j x^{j-1} - \sum_{j=0}^{m-1} j(j-1)\, a_j x^{j-1} \tag{7.15}$$

$$= \sum_{j=0}^{m-1} j(m-j)\, a_j x^{j-1} > 0. \quad \blacksquare \tag{7.16}$$

Lemma 7.17. *For $q > 1$, $r_0(z)$ achieves its minimum value at $\pm\beta^*$, $0 < \beta^* < 1$, and nowhere else. Furthermore,*

$$r_0''(\beta^*) > 0. \tag{7.18}$$

For $q = 1$, we have

$$r_0(z) = z^2 - \tfrac{1}{2}. \tag{7.19}$$

Proof. Let $q > 1$. First, observe from (7.9) that Lemma 7.11 applies to $r_0(z)$. Now suppose that z_1 and z_2 are consecutive positive roots of $r_0'(z)$. Then we have $r_0''(z_1) > 0$ and $r_0''(z_2) > 0$, which implies that z_1 and z_2 cannot be consecutive roots of $r_0'(z)$. Thus there exists at most one positive root of the equation $r_0'(z) = 0$.

However, it is clear from (7.9) that

$$r_0'(z) < 0 \qquad \text{for} \quad 0 < z < \eta \tag{7.20}$$

for η sufficiently small, and clearly

$$r_0'(1) > 0. \tag{7.21}$$

Thus there exists $\beta^* \in (0, 1)$ such that $r_0'(\beta^*) = 0$. We conclude that $r_0'(z)$ vanishes for $z = 0$, $\pm\beta^*$, and nowhere else. This proves the Lemma in the case $q > 1$.

On the other hand formula (7.19) follows from formula (7.9) with $q = 1$. ∎

Remark 7.22. For future reference we note that by Lemma 7.17, the graph of $r_0(z)$ is as shown in Fig. 8, which is drawn in the case $q = 2$. The graphs for $q > 2$ are, of course, similar.

Lemma 7.23. *Let ψ be given as in* (7.7).

Then for $q > 1$, there exists $t_{+,2q}$ such that for $0 < t \leqslant t_{+,2q}$, $\psi(y) \geqslant 0$ for all $y \in [-1, 1]$ and for $t_{+,2q} < t < t_{+,2q} + \varepsilon$, $\varepsilon > 0$ sufficiently small, there are two open intervals on which $\psi < 0$.

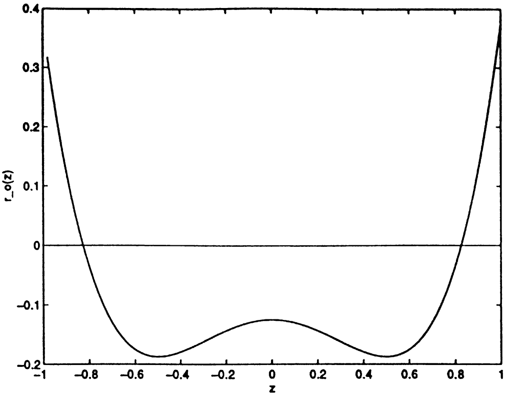

FIG. 8. Plot of $r_0(z)$ for $q=2$. Note two minima (in plot, at locations $\pm\beta^* = \pm\frac{1}{2}$).

For $q=1$, we have $t_{+,2q}=2$. That is, if $t\leqslant 2$, then $\psi(y)\geqslant 0$ for all $y\in[-1,1]$. If $t>2$, then $\psi(y)<0$ for y in a neighborhood of $y=0$.

Proof. The case $q=1$ follows explicitly from the formula for ψ:

$$\psi = \frac{i}{\pi(z^2-1)^{1/2}}\left[1+t\left(z^2-\frac{1}{2}\right)\right]. \tag{7.24}$$

For $q>1$, let $t_{+,2q}$ solve

$$1+qt_{+,2q}r_0(\beta^*)=0. \tag{7.25}$$

Then it is clear that for $t<t_{+,2q}$, we have

$$1+qtr_0(z)>0 \qquad \text{for all} \quad z\in(-1,1) \tag{7.26}$$

and hence, form (7.7),

$$\psi(z)\geqslant 0 \qquad \text{for all} \quad z\in(-1,1). \tag{7.27}$$

Furthermore, for $t_{+,2q}<t<t_{+,2q}+\varepsilon$, we have the existence of two regions where $1+qt_{+,2q}r_0(z)<0$, and hence ψ will be negative on those regions as well. ∎

To construct the maximizer for $t>t_{+,2q}$, we follow examples from the integrable systems literature (see, for example, [20]), and increase the number of intervals in the support J of the maximizer.

Thus we suppose that for $q>1$,

$$J=(-1,-\eta_2)\cup(-\eta_1,\eta_1)\cup(\eta_2,1) \tag{7.28}$$

and for $q=1$,

$$J=(-1,\ -\eta)\cup(\eta,\ 1), \tag{7.29}$$

Thus in the notation of Theorem 1.38, $\alpha_0=-1$, $\beta_0=-\eta_2$, $\alpha_1=-\eta_1$, etc. From Theorem 1.38, we must have

$$\psi = ReF_{+}, \tag{7.30}$$

where

$$F(z)=\frac{R(z)^{1/2}}{\pi i}\int_J \frac{-iqt}{\pi}\frac{y^{2q-1}}{R(y)^{1/2}}\frac{dy}{y-z}, \tag{7.31}$$

and for $q>1$,

$$R(z)=\frac{(z^2-\eta_1^2)(z^2-\eta_2^2)}{z^2-1}. \tag{7.32}$$

As $2q$ is even, and as $R(y)^{1/2}=R(y)_+^{1/2}$ is even on J, $F(z)$ decays at ∞. For the case $q=1$, we have

$$F=\left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2}\frac{1}{\pi i}\int_J\frac{-it}{\pi}\frac{y}{y-z}\left(\frac{y^2-1}{y^2-\eta^2}\right)^{1/2}dy, \tag{7.33}$$

which also decays as $z\to\infty$.

The parameters η_1 and η_2 in (7.31) and η in (7.33) must now be determined. We have the condition

$$\int\psi=1, \tag{7.34}$$

which is sufficient to determine η in (7.33) for the case $q=1$, but this is of course not enough information to determine both η_1 and η_2 in (7.31). A second condition arises from the fact that the Lagrange multiplier ℓ must be the same in all intervals comprising J (see equation (3.10) above), and so we have

$$\int_{\eta_1}^{\eta_2}\left(H\psi+\frac{qty^{2q-1}}{\pi}\right)dy=0 \tag{7.35}$$

The companion relation $\int_{-\eta_2}^{-\eta_1}(H\psi+qt(y^{2q-1}/\pi))\,dy=0$ follows by symmetry once (7.35) is established.

Assuming for the moment that we can solve equations (7.34) and (7.35) for η_1 and η_2 (or equation (7.34) for η if $q=1$), the quantity ψ defined by (7.30) above will be the solution of the maximization problem if we can establish that ψ also satisfies conditions (5.7)–(5.10).

We will consider the case of general $q \geqslant 2$ first. Thus we must prove that the equations (7.34) and (7.35) can be solved simultaneously for (η_1, η_2), say for $0 < t - t_{+,2q} < \varepsilon$ (for some small $\varepsilon > 0$). Once this is established, we must then show that ψ defined by (7.30) with η_1 and η_2 chosen to solve (7.34) and (7.35) satisfies the inequalities

$$\begin{gathered} \psi(y) > 0 \qquad \text{for all} \quad y \in J \\ \int_{\eta_1}^{\lambda} \left(H\psi + \frac{qt}{\pi}\, y^{2q-1} \right) dy < 0 \qquad \forall \lambda \in (\eta_1, \eta_2). \end{gathered} \tag{7.36}$$

Again the companion inequality $\int_{-\eta_2}^{\lambda} (H\psi + (qt/\pi)\, y^{2q-1})\, dy < 0$ for $\lambda \in (-\eta_2, -\eta_1)$ follows from (7.36) by symmetry.

This type of problem appears in the analysis of limits of completely integrable dynamical systems [6, 21, 32, 37]. There, the endpoints of the support are functions of space x and time t, and they evolve according to a hyperbolic system of equations, which are particular examples of the so-called Whitham equations. The number of intervals in the support is fixed unless one or more of the quantities $\eta_j(x, t)$ develops a shock. Beyond the time of shock formation, an additional interval forms in the support and the number of endpoints $\eta_j(x, t)$ increases by two. This was first established by Tian [32], and Wright [37] for the case of the zero dispersion limit of the KdV equation. Their method consisted of a detailed analysis of the Whitham averaged equations in a neighborhood of the shock location. Here, we use methods developed in [6] for the study of the continuum limit of the Toda lattice to analyze this connection, or phase transition, problem.

We begin with some definitions. Using formula (7.31), and exchanging the order of integration, the condition $\int \psi = 1$ can be re-written

$$T_{2q} \equiv \frac{-qt}{2\pi i} \oint_{\Gamma_\infty} \frac{y^{2q}}{R(y)^{1/2}}\, dy = 1, \tag{7.37}$$

where Γ_∞ is a counterclockwise contour on a large circle containing J. Set

$$A \equiv \int_{\eta_1}^{\eta_2} \left(H\psi + \frac{qt}{\pi}\, y^{2q-1} \right) dy. \tag{7.38}$$

Then (7.35) takes the form

$$A = 0. \tag{7.39}$$

LEMMA 7.40. *For $0<t-t_{+,2q}<\varepsilon$, $\varepsilon>0$ small, there exists a solution (η_1, η_2), $0<\eta_1<\eta_2$ to the set of equations* (7.37, 7.39).

Proof. For η_1, η_2 near $\eta^*=\beta^*$, introduce the following quantities:

$$b_1=\eta_1^2-(\eta^*)^2, \tag{7.41}$$

$$b_2=\eta_2^2-(\eta^*)^2, \tag{7.42}$$

$$T(b_1, b_2)=T_{2q}(\eta_1, \eta_2). \tag{7.43}$$

Our first goal is to establish that T satisfies

$$T(0, 0)=\frac{t}{t_{+,2q}}, \tag{7.44}$$

$$T_{b_j}(0, 0)=0 \qquad \text{for} \quad j=1, 2, \tag{7.45}$$

$$T_{b_ib_i}(0, 0)=\frac{-3qt}{8}\left[\left(\frac{1}{2\eta^*}\right)^2 r_0''(\eta^*)\right] \qquad \text{for} \quad i=1, 2, \tag{7.46}$$

$$T_{b_1b_2}(0, 0)=\frac{-qt}{8}\left[\left(\frac{1}{2\eta^*}\right)^2 r_0''(\eta^*)\right]. \tag{7.47}$$

Indeed, T_{2q}, defined in (7.37), is an integral over a contour well away from the branch points $\pm\eta_1$, $\pm\eta_2$, and hence the function T can be differentiated with respect to b_1 or b_2 by differentiating under the integral sign. We thus have the following formulae:

$$T_{b_i}=\frac{-qt}{2\pi i}\oint_{\Gamma_\infty}\frac{y^{2q}}{R(y)^{1/2}}\frac{1/2}{y^2-(b_i+(\eta^*)^2)}\,dy, \tag{7.48}$$

$$T_{b_ib_i}=\frac{-qt}{2\pi i}\oint_{\Gamma_\infty}\frac{y^{2q}}{R(y)^{1/2}}\frac{3/4}{(y^2-(b_i+(\eta^*)^2))^2}\,dy, \tag{7.49}$$

$$T_{b_ib_2}=\frac{-qt}{2\pi i}\oint_{\Gamma_\infty}\frac{y^{2q}}{R(y)^{1/2}}\frac{1/4}{(y^2-(b_1+(\eta^*)^2))(y^2-(b_2+(\eta^*)^2))}\,dy. \tag{7.50}$$

These formulae may be evaluated at $b_i=0$:

$$\begin{aligned}T_{b_i}(0, 0)&=\frac{-qt}{2\pi i}\oint_{\Gamma_\infty} y^{2q}(y^2-1)^{1/2}\frac{1/2}{(y^2-(\eta^*)^2)^2}\,dy\\ &=-qt\frac{1}{4\eta^*}\frac{d}{dz}\bigg|_{z=\eta^*} r_0(z)=0,\end{aligned} \tag{7.51}$$

$$T_{b_i b_i}(0,0) = \frac{-qt}{2\pi i} \oint_{\Gamma_\infty} y^{2q}(y^2-1)^{1/2} \frac{3/4}{(y^2-(\eta^*)^2)^3}\, dy$$

$$= \frac{-3qt}{4} \frac{1}{8(\eta^*)^2} \frac{d^2}{dz^2}\bigg|_{z=\eta^*} r_0(z), \tag{7.52}$$

$$T_{b_i b_2}(0,0) = \frac{-qt}{2\pi i} \oint_{\Gamma_\infty} y^{2q}(y^2-1)^{1/2} \frac{1/4}{(y^2-(\eta^*)^2)^3}\, dy$$

$$= \frac{-qt}{4} \frac{1}{8(\eta^*)^2} \frac{d^2}{dz^2}\bigg|_{z=\eta^*} r_0(z). \tag{7.51}$$

This establishes the validity of formulae (7.45)–(7.47). To prove (7.44), simply insert (7.32) with $\eta_1 = \eta_2 = \eta^*$, into (7.37), and compare with (7.10) and (7.25).

As we know that

$$\frac{d^2}{dz^2}\bigg|_{z=\eta^*} r_0(z) > 0, \tag{7.54}$$

(7.46) and (7.47) show that for (b_1, b_2) near (0, 0),

$$\begin{pmatrix} T_{b_1 b_1} & T_{b_1 b_2} \\ T_{b_1 b_2} & T_{b_2 b_2} \end{pmatrix} \tag{7.44}$$

is strictly negative definite and hence T is locally a paraboloid. More precisely, using (7.44),

$$T = \frac{t}{t_{+,2q}} - \left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad P \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right) + O((b_1^2 + b_2^2)^{3/2}) \tag{7.56}$$

for some strictly positive definite matrix P. Using polar coordinates, an elementary argument now shows that for $0 < t - t_{+,2q} < \varepsilon$, $\varepsilon > 0$ small,

$$T(b_1, b_2) = 1 \tag{7.57}$$

has a continuous 1-parameter family of solutions $b_1 = r(\theta)\cos\theta$, $b_2 = r(\theta)\sin\theta$, $r(\theta) > 0$, $0 \leqslant \theta \leqslant 2\pi$.

Now we recall the representation (1.44) of the function F,

$$F(z) = -i \frac{V'(z)}{2\pi} + \frac{i}{2\pi} R(z)^{1/2}\, \hat{r}(z), \tag{7.58}$$

where

$$\hat{r}(z) = \frac{1}{2\pi i} \oint \frac{2qty^{2q-1}}{R(y)^{1/2}} \frac{dy}{y-z} \tag{7.59}$$

$$= \frac{1}{2\pi i} \oint \frac{2qty^{2q}}{R(y)^{1/2}} \frac{dy}{y^2-z^2} \tag{7.60}$$

Formula (7.60) follows from (7.59) by symmetry, $\hat{r}(z) = \hat{r}(-z)$. As $F_+ = \psi + iH\psi$, we see that for $\eta_1 < y < \eta_2$,

$$H\psi + \frac{qt}{\pi} y^{2q-1} = \frac{1}{2\pi} R^{1/2}\hat{r}. \tag{7.61}$$

To prove that $A(b_1, b_2) = 0$ for some $b_1 < b_2$ on the circle $T(b_1, b_2) = 1$, we proceed as follows. First we show that for $y \in (\eta_1, \eta_2)$,

$$R(y)^{1/2}\, \hat{r}(y) < 0 \tag{7.62}$$

for $b_1 < b_2 = 0$ on $T(b_1, 0) = 1$, and hence $A(b_1, 0) < 0$ by (7.38) and (7.61). Then we show that on (η_1, η_2)

$$R(y)^{1/2}\, \hat{r}(y) > 0 \tag{7.63}$$

for $b_1 = 0 < b_2$, $T(0, b_2) = 1$, and hence $A(0, b_2) > 0$, again by (7.38) and (7.61). The result then follows by continuity.

Now evaluating $\hat{r}$ for $b_1 = b_2 = 0$, we have

$$\hat{r}(z) = \frac{1}{2\pi i} \oint 2qty^{2q}(y^2-1)^{1/2} \frac{dy}{(y^2-z^2)(y^2-(\eta^*)^2)} \tag{7.64}$$

$$= 2qt \frac{r_0(z) - r_0(\eta^*)}{(z^2-(\eta^*)^2)}, \tag{7.65}$$

where we have used the representation (7.10) of the function $r_0(z)$.

From (7.65) and (the $q \geqslant 2$ analog of) Fig. 8, we see that $\hat{r}(z; b_1 = 0, b_2 = 0)$ has precisely two real, simple roots $z = \pm\eta^*$. (The simplicity of the roots follows from the positivity of $r_0''(\pm\eta^*)$.) We claim that for (b_1, b_2) small, $\hat{r}(z; b_1, b_2)$ also has precisely two roots $z_\pm(b_1, b_2)$, where $z_\pm$ is close to $\pm\eta^*$. Indeed, evaluating (7.60) by residues at ∞, we see that

$$\hat{r}(z; b_1, b_2) = 2qtz^{2q-2} + \hat{a}_{2q-4} z^{2q-4} + \cdots + \hat{a}_0 \tag{7.66}$$

where the coefficients $\hat{a}_{2j}$ are real analytic functions of (b_1, b_2). It follows from (7.66) that there exists $L > 0$, independent of (b_1, b_2), such that $\hat{r}(z)$ has no real zeros for $|z| \geqslant L$ and for all (b_1, b_2) small. Now $\hat{r}(z; b_1 = 0,$

$b_2 = 0$) has precisely two roots in the rectangular box $B_{L,\delta} = \{|Re\, z| \leqslant L, |Im\, z| \leqslant \delta\}$ for some (small) $\delta > 0$. Hence by Rouche's Theorem, $\hat{r}(z; b_1, b_2)$ has precisely two roots $z_+(b_1, b_2) \sim \eta^*$, $z_-(b_1, b_2) \sim -\eta^*$ inside $B_{L,\delta}$ for all (b_1, b_2) small. Moreover, as $\hat{r}(z; b_1, b_2)$ has real coefficients, z_+ and z_- must be real. On the other hand we know that $\hat{r}(z; b_1, b_2)$ has no real roots for $|z| \geqslant L$, and we conclude that

Property 7.67. $\hat{r}(z; b_1, b_2)$ has precisely two real roots $z_+ > z_-$ for all (b_1, b_2) small.

Now we show that for b_1 small and negative, and $b_2 = 0$,

$$\eta_1 < \eta^* = \eta_2 < z_+(b_1, 0). \tag{7.68}$$

Differentiating the equation $\hat{r}(z_+, b_1, 0) = 0$ at $b_1 = 0$, we obtain

$$\hat{r}_z(\eta^*; 0, 0)\frac{\partial z_+}{\partial b_1} + \hat{r}_{b_1}(\eta^*; 0, 0) = 0. \tag{7.69}$$

but for $b_1 = b_2 = 0$,

$$\begin{aligned} \hat{r}'(\eta^*) &= \frac{1}{2\pi i}\oint 2qty^{2q}(y^2-1)^{1/2}\frac{2\eta^*}{(y^2-(\eta^*)^2)^3}\,dy \\ &= \frac{qt}{2\eta^*}\, r_0''(\eta^*) > 0, \end{aligned} \tag{7.70}$$

and

$$\left(\frac{\partial}{\partial b_j}\hat{r}\right)(\eta^*) = \frac{qt}{8(\eta^*)^2}\, r_0''(\eta^*) > 0. \tag{7.71}$$

Hence $(\partial z_+/\partial b_1)(0, 0) < 0$, and (7.68) follows for b_1 small and negative. A similar argument shows that for b_2 small and positive, and $b_1 = 0$,

$$z_+(0, b_2) < \eta_1 = \eta^* < \eta_2. \tag{7.72}$$

Since $R(y)^{1/2} > 0$, (7.68) and (7.72) imply (7.62) and (7.63), and the proof of Lemma 7.40 is complete. ∎

Remark. As $\hat{r}(z) = \hat{r}(-z)$, we must of course have $z_+(b_1, b_2) = -z_-(b_1, b_2)$.

For η_1 and η_2 in the above Lemma, we have $A(\eta_1, \eta_2) = 0$ and hence by (7.61) we conclude that necessarily $z_+(b_1, b_2) \in (\eta_1, \eta_2)$ and $z_-(b_1, b_2) \in (-\eta_2, \eta_1)$. In particular this means that $\hat{r}(y)$ is nonzero on J. But from (7.58), we have for $y \in J$,

$$\psi(y) = \frac{i}{2\pi} (R(y)^{1/2})_+ \hat{r}(y) \tag{7.73}$$

and hence

$$\psi(y) > 0 \qquad \text{on } J. \tag{7.74}$$

To complete the proof that $\psi = Re\, F_+$ is indeed the maximizer we must still show that

$$G(\lambda) \equiv \int_{\eta_1}^{\lambda} \left(H\psi + \frac{qt}{\pi} y^{2q-1} \right) dy < 0 \qquad \text{for} \quad \lambda \in (\eta_1, \eta_2). \tag{7.75}$$

To see this, note

$$G(\eta_1) = 0$$

$$G(\eta_2) = 0$$

and as $\hat{r}(\eta_1) < 0$,

$$G(\lambda) < 0$$

for $0 < \lambda - \eta_1 < \varepsilon$, ε small, by (7.61). Clearly if $G(\lambda) > 0$ for some $\eta_1 < \lambda < \eta_2$, then there would be at least two points $\eta_1 < \lambda_1 < \lambda_2 < \eta_2$ for which $G'(\lambda_i) = 0$. But $G'(\lambda) = (1/2\pi)\, R(\lambda)^{1/2}\, \hat{r}(\lambda)$, which has only one root in (η_1, η_2), and this is a contradiction. Thus $G(\lambda) < 0$ for all $\lambda \in (\eta_1, \eta_2)$.

Remark 7.76. The reader will notice that we have not addressed the question of uniqueness for the solution (η_1, η_2) of (7.37), (7.39). In this connection, we recall from Problem 1 that once the equilibrium measure is constructed, it is *necessarily* unique. However, for $0 < t - t_{+,2q} < \varepsilon$, $\varepsilon > 0$ small, we have seen that for any solution (η_1, η_2) of (7.37), (7.39), the associated measure $\psi(x)\, dx$ defined by (7.30), (7.31) necessarily satisfies all the variational conditions (5.7)–(5.10). In particular ψ must be the unique equilibrium measure, and this proves in turn the uniqueness of the solution (η_1, η_2) of (7.37), (7.39) for $0 < t - t_{+,2q} < \varepsilon$.

We have now established $\mathbf{C}_4$, $\mathbf{C}_3$, $\mathbf{B}_3$, and $\mathbf{A}_2$ of Theorem 1.52.

We return now to the case $q=1$. From (7.33) for $t \geqslant 2$, we have

$$F(z)=\left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2} \frac{1}{\pi i}\int_J \frac{-it}{\pi}\, y\left(\frac{y^2-1}{y^2-\eta^2}\right)^{1/2} \frac{dy}{y-z} \tag{7.77}$$

$$=\frac{-it}{\pi}\left[z-z\left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2}\right]. \tag{7.78}$$

The condition $\int \psi=1$ may again be evaluated explicitly, and we find

$$\frac{t(1-\eta^2)}{2}=1, \tag{7.79}$$

from which the quantity η can clearly be determined, for all $t \geqslant 2$. The integrand on the left hand side of (5.8) can again be evaluated by inserting (7.78) into the relation $F_+=\psi+iH\psi$, and (5.8) becomes

$$\frac{t}{\pi}\int_{-\eta}^{\lambda} z\left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2} dz<0, \tag{7.80}$$

for all $\lambda \in (-\eta, \eta)$, which is obviously true. Finally for $y \in J$,

$$\psi(y)=\frac{it}{\pi}\, y\left(\frac{y^2-\eta^2}{y^2-1}\right)_+^{1/2} \tag{7.81}$$

and so $\psi(y) \geqslant 0$. Thus $\psi=Re\, F_+$ is indeed the maximizer. Furthermore, as $t \to \infty$, we have $\eta \to 1$.

This completes the proof of case A of Theorem 1.52.

We now consider the case $q=2$. In this case, $r_0(z)=z^4-(z^2/2+1/8)$, by (7.9), and so $\eta^*=1/2$, and hence $r_0(\eta^*)=-3/16$. From (7.25) we learn in turn that $t_{+,4}=t_{+,4}^{(2)}=8/3$.

For $8/3<t<8/3+\varepsilon$, ($\varepsilon>0$ and small) the solution of the maximization problem is given by $\psi=Re\, F_+$, with F as in (7.31). Deforming the contour as before we obtain

$$F(z)+\frac{2it}{\pi} z^3=\frac{2it}{\pi} R(z)^{1/2} h_4(z), \tag{7.82}$$

where

$$h_4(z) = \frac{1}{2\pi i}\oint_{\Gamma_\infty} \frac{y^3}{R(y)^{1/2}}\frac{dy}{y-z} \tag{7.83}$$

$$= \frac{1}{2\pi i}\oint_{\Gamma_\infty} \frac{y^4}{R(y)^{1/2}}\frac{dy}{y^2-z^2} \tag{7.84}$$

$$= z^2 + \frac{\eta_1^2+\eta_2^2-1}{2}, \tag{7.85}$$

where in the first line of (7.83)

$$R(z) = \frac{(z^2-\eta_1^2)(z^2-\eta_2^2)}{z^2-1}.$$

As above, the parameters η_1 and η_2 are determined by the system of equations (7.37), (7.39). Again Γ_∞ is a large counterclockwise contour containing J and z.

In what follows we shall describe the solution to the maximization problem for all $t > 8/3$. We will show that there is a second critical value $t^{(i)}_{+,4}$; for all $t^{(2)}_{+,4} < t < t^{(1)}_{+,4}$, J consists of three bands $(-1, -\eta_2) \cup (-\eta_1, \eta_1) \cup (\eta_2, 1)$, and as $t \uparrow t^{(1)}_{+,4}$, $\eta_1 \downarrow 0$, i.e., the central band vanishes. Then we will show that for all $t > t^{(1)}_{+,4}$, the maximization problem is solved by the simpler ansatz $J = (-1, -\eta) \cup (\eta, 1)$.

The first step is to show that there is no local obstruction to increasing t. More precisely, we make the following Claim. Set (cf. (7.37))

$$T(\eta_1, \eta_2) = T_4(\eta_1, \eta_2) = -\frac{t}{\pi i}\oint_{\Gamma_\infty} \frac{y^4}{R(y)^{1/2}}\, dy, \tag{7.86}$$

$$A(\eta_1, \eta_2) = \frac{2t}{\pi}\int_{\eta_1}^{\eta_2} R(s)^{1/2}\, h_4(s)\, ds$$

$$= \int_{\eta_1}^{\eta_2} \left(\frac{F_+(s)}{i} + \frac{2ts^3}{\pi}\right) ds. \tag{7.87}$$

Both these formulae for $A(\eta_1, \eta_2)$ are obtained from (7.38) using $F_+ = \psi + iH\psi$ and the explicit formula (7.82) for F. Note that T, A and F are all C^1 functions of η_1^2 and η_2^2 for $0 < \eta_1 < \eta_2 < 1$.

Claim. *For $0 < t - 8/3 < \varepsilon$, $\varepsilon > 0$ small, the Jacobian of the transformation $(\eta_1, \eta_2) \mapsto (T, A)$ is nonzero on the solution to the system* (7.37), (7.39).

Proof. We first establish the following relationships:

$$T_{\eta_j^2} = -th_4(\eta_j), \tag{7.88}$$

$$F_{\eta_j^2} = -\frac{T_{\eta_j^2}}{\pi i} \frac{R(z)^{1/2}}{z^2 - \eta_j^2}, \tag{7.89}$$

$$A_{\eta_j^2} = \int_{\eta_1}^{\eta_2} \frac{T_{\eta_j^2}}{\pi} \frac{R(z)^{1/2}}{z^2 - \eta_j^2}\, dz. \tag{7.90}$$

From (7.86) above, we compute

$$\begin{aligned} T_{\eta_j^2} &= \frac{-t}{\pi i} \oint_{\Gamma_\infty} \frac{y^4}{R(y)^{1/2}} \frac{1/2}{y^2 - \eta_j^2}\, dy \\ &= -th_4(\eta_j), \end{aligned} \tag{7.91}$$

which establishes (7.88).

Equation (7.89) can be proved directly by differentiating formula (7.82) and using (7.85), (7.88). To compute $\partial A / \partial n_j^2$, note that $R(s)^{1/2} h_4(s)$ vanishes at $s = \eta_1$ and $s = \eta_2$, and hence

$$\frac{\partial A}{\partial \eta_j^2} = \frac{1}{i} \int_{\eta_1}^{\eta_2} \frac{\partial F_+}{\partial \eta_j^2}(s)\, ds$$

Formula (7.90) now follows from (7.89).

Using formulae (7.88)–(7.90), we obtain

$$\begin{aligned} |J| &= \frac{T_{\eta_1^2} T_{\eta_2^2}}{\pi} \begin{vmatrix} 1 & 1 \\ \int \dfrac{R^{1/2}}{z^2 - \eta_1^2} & \int \dfrac{R^{1/2}}{z^2 - \eta_2^2} \end{vmatrix} \\ &= \frac{T_{\eta_1^2} T_{\eta_2^2} (\eta_2^2 - \eta_1^2)}{\pi} \int_{\eta_1}^{\eta_2} \frac{dz}{((z^2 - \eta_1^2)(z^2 - \eta_2^2)(z^2 - 1))^{1/2}} \end{aligned}$$

Now, since $\eta_1 < \eta_2$, and $A = 0$, it follows as before that the two roots of the function $h_4(z)$ must lie one in each of the two intervals $(-\eta_2, -\eta_1)$, (η_1, η_2). Hence, from (7.88), we see immediately that on a solution to the system (7.37), (7.39),

$$T_{\eta_1^2} > 0 \qquad \text{and} \qquad T_{\eta_2^2} < 0. \tag{7.93}$$

Thus, from (7.92), we see that the Jacobian of the transformation $(\eta_1, \eta_2) \mapsto (T, A)$ is nonzero on the solution to (7.37)–(7.39). ∎

The preceding calculations show that for $0 < t - t^{(2)}_{+,4} < \varepsilon$, $\varepsilon > 0$ and small, equations (7.37) and (7.39) have solutions $0 < \eta_1(t) < \eta_2(t) < 1$, which give rise to the equilibrium measure $\psi(x)\,dx$ via (7.30) and (7.31). As noted in Remark 7.76, this implies that $\eta_1(t)$ and $\eta_2(t)$ are the unique solutions of (7.37) and (7.39). However, in the case $q = 2$, we make the following claim.

CLAIM. *If* (η_1, η_2) *solve* $T(\eta_1, \eta_2) = 1$, $A(\eta_1, \eta_2) = 0$, $0 < \eta_1 < \eta_2 < 1$, *for* any $t > 0$ *then necessarily* $\psi(x)\,dx$ *constructed via* (7.30) *and* (7.31), *is the equilibrium measure, i.e. the variational inequalities* (5.8)–(5.10) *are automatically satisfied. Also, as noted above,* η_1 *and* η_2 *are unique.*

Indeed, for general $g \geqslant 2$, the crux of the proof given above that the inequalities (5.8)–(5.10) are satisfied for $0 < t - t_{+,2q} < \varepsilon$, is that $\hat{r}(z)$ in (7.58)–(7.60) has precisely two real roots. However for $q = 2$, we have

$$\hat{r}(z) = 4th_4(z) = 4t\left(z^2 + \frac{\eta_1^2 + \eta_2^2 - 1}{2}\right),$$

which clearly has two roots. The proof of the above Claim is now clear.

Set

$$S = \left\{ t\colon \begin{array}{l} \text{for all } s \in (t^{(2)}_{+,4}, t], \text{ equations (7.37), (7.39)} \\ \text{have a solution } 0 < \eta_1(s) < \eta_2(s) < 1 \end{array} \right\}. \tag{7.94}$$

By the preceding remarks, $(\eta_1(s), \eta_2(s))$ are the unique solutions of $T(\eta_1, \eta_2) = 1$, $A(\eta_1, \eta_2) = 0$, $t^{(2)}_{+,4} < s < t$, $t \in S$. By the Claim following (7.87), $\eta_1(s)$ and $\eta_2(s)$ are necessarily continuously differentiable. Also the Claim implies that S is an open set. On the other hand, S is clearly a non-empty interval and so we have the representation

$$S = (t^{(2)}_{+,2}, \hat{t}) \tag{7.95}$$

for some $t^{(2)}_{+,4} < \hat{t} \leqslant \infty$.

We show first that $\hat{t} < \infty$. To accomplish this, observe from (7.82) that for $0 < \eta_1 < \eta_2 < 1$,

$$F(z) = \frac{-2it}{\pi} z^3 + \frac{2it}{\pi}\left(\frac{(z^2 - \eta_1^2)(z^2 - \eta_2^2)}{z^2 - 1}\right)^{1/2} \times \left(z^2 + \frac{\eta_1^2 + \eta_2^2 - 1}{2}\right), \tag{7.96}$$

which has, in particular, a continuous extension to $0 \leqslant \eta_1 < \eta_2 \leqslant 1$. The same is clearly true for

$$T = \frac{-t}{\pi i} \oint_{\Gamma_\infty} y^4 \left(\frac{y^2 - 1}{(y^2 - \eta_1^2)(y^2 - \eta_2^2)} \right)^{1/2} dy$$

and also for

$$A = \frac{2t}{\pi} \int_{\eta_1}^{\eta_2} \left(\frac{(s^2 - \eta_1^2)(s^2 - \eta_2^2)}{s^2 - 1} \right)^{1/2} \left(s^2 + \frac{\eta_1^2 + \eta_2^2 - 1}{2} \right) ds.$$

Furthermore, it is clear from the preceding comments and calculations that if

$$T(\eta_1 = 0, \eta_2, t) = 1,$$

$$A(\eta_1 = 0, \eta_2, t) = 0,$$

for some $0 < \eta_2 \leqslant 1$ and for some $t = t^0 > t^{(2)}_{+,4}$, then $\psi(x)\, dx = (ReF_+(x))\, dx$ constructed from (7.96) is the unique equilibrium measure for $t = t^0$. It then follows that $\hat{t} \leqslant t^0 < \infty$.

We have

$$A(\eta_1 = 0, \eta_2) = \int_0^{\eta_2} \frac{2t}{\pi} \left(\frac{z^2 - \eta_2^2}{z^2 - 1} \right)^{1/2} \left(z^2 - \frac{1 - \eta_2^2}{2} \right) z\, dz$$

$$= \int_0^{\eta_2^2} \frac{t}{\pi} \left(\frac{u - \eta_2^2}{u - 1} \right)^{1/2} \left(u - \frac{1 - \eta_2^2}{2} \right) du. \tag{7.97}$$

Now $A(0, 1) = t/2\pi$, and $A(0, 1/\sqrt{3}) < 0$. Thus there is at least one value $\eta_2 = \eta^\# \in (1/\sqrt{3}, 1)$ for which $A(0, \eta^\#) = 0$. Now observe by a residue calculation that

$$T(0, \eta^\#) = \frac{-t}{\pi i} \oint_{\Gamma_\infty} y^3 \left(\frac{y^2 - 1}{y^2 - (\eta^\#)^2} \right)^{1/2} dy = \frac{t}{4} (1 - (\eta^\#)^2)(1 + 3(\eta^\#)^2),$$

and we may chose a finite value $t = t^0 > 0$ so that $T(0, \eta\#) = 1$. It follows, in particular, that the support of the equilibrium measure for $t = t^0$ consists of two intervals, $(-1, -\eta^\#) \cup (\eta^\#, 1)$, and hence $t^0 > t^{(2)}_{+,4}$. But then $t^{(2)}_{+,4} < \hat{t} \leqslant t^0 < \infty$.

Next we show that as $t \uparrow \hat{t}$, $\eta_1(t) \downarrow 0$. For any $t^{(2)}_{+,4} < t < \hat{t}$, we have the unique solutions $0 < \eta_1 < \eta_2 < 1$ of (7.37), (7.39). Differentiating the equations $T(\eta_1, \eta_2, t) = 1$, $A(\eta_1, \eta_2, t) = 0$ with respect to t, and using (7.90), we obtain the following formulae for $\partial \eta_1^2 / \partial t$, $\partial \eta_2^2 / \partial t$,

$$T_{\eta_1^2}\frac{\partial n_1^2}{\partial t}=\frac{-\int_{\eta_1}^{\eta_2}\frac{R^{1/2}}{z^2-\eta_2^2}\,dz}{t(\eta_2^2-\eta_1^2)\int_{\eta_1}^{\eta_2}\frac{dz}{[(z^2-\eta_1^2)(z^2-\eta_2^2)(z^2-1)]^{1/2}}}<0, \tag{7.98}$$

$$T_{\eta_2^2}\frac{\partial n_2^2}{\partial t}=\frac{\int_{\eta_1}^{\eta_2}\frac{R^{1/2}}{z^2-\eta_2^2}\,dz}{t(\eta_2^2-\eta_1^2)\int_{\eta_1}^{\eta_2}\frac{dz}{[(z^2-\eta_1^2)(z^2-\eta_2^2)(z^2-1)]^{1/2}}}<0. \tag{7.99}$$

But then from (7.93) we learn that $\partial\eta_1^2/\partial t<0$ and $\partial\eta_2^2/\partial t>0$. In particular we learn that the limits

$$\hat{\eta}_1=\lim_{t\uparrow\hat{t}}\eta_1(t), \qquad \hat{\eta}_2=\lim_{t\uparrow\hat{t}}\eta_2(t) \tag{7.100}$$

exist and

$$0\leqslant\hat{\eta}_1<\hat{\eta}_2\leqslant 1. \tag{7.101}$$

Now we cannot have $\hat{\eta}=1$, as this would contradict Proposition 4.18(ii), or (iii). Alternatively we can argue directly as follows. If $\hat{\eta}_2=1$, then taking the limit $t\uparrow\hat{t}$ in $T(\eta_1(t),\eta_2(t),t)=1$, we find

$$1=\frac{-\hat{t}}{\pi i}\oint_{\Gamma_\infty}\frac{y^4}{(y^2-\hat{\eta}_1^2)^{1/2}}\,dy=\frac{-3\hat{t}\hat{\eta}_1^4}{4},$$

which is a contradiction. Hence $\hat{\eta}_2<1$. It then follows that $\hat{\eta}_1=0$; otherwise we could extend the solution to $T=1$, $A=0$ beyond $\hat{t}$, which is not possible. Thus as $t\uparrow\hat{t}$, $\eta_1(t)\downarrow\hat{\eta}_1=0$, $\eta_2(t)\uparrow\hat{\eta}_2<1$.

Set $t_{+,4}^{(1)}=\hat{t}$. We complete the proof of case B of Theorem 1.52 by showing that for all $t>t_{+,4}^{(1)}$, the support of the equilibrium measure consists of two intervals $J=(-1,-\eta)\cup(\eta,1)$, $0<\eta<1$. From Theorem (1.38) we must have $\psi=Re\,F_+$, where

$$F(z)=\frac{R(z)^{1/2}}{\pi i}\int_J\frac{-2it}{\pi}\frac{y^3}{R(y)^{1/2}}\frac{dy}{y-z}, \tag{7.102}$$

and

$$R(z)=\left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2}. \tag{7.103}$$

As before we also have the representation

$$F(z)+\frac{2it}{\pi}z^3=\frac{2it}{\pi}R(z)^{1/2}\,\hat{h}_4(z), \tag{7.104}$$

where

$$\begin{aligned}\hat{h}_4(z)&=\frac{1}{2\pi i}\oint_{\Gamma_\infty}\frac{y^3}{R(y)^{1/2}}\frac{dy}{y-z}\\&=z\left(z^2-\frac{1-\eta^2}{2}\right).\end{aligned} \tag{7.105}$$

Clearly F defined by (7.102) decays as $z\to\infty$.

The condition $\int\psi=1$ is expressed by the formula

$$\hat{T}\equiv\frac{-t}{\pi i}\oint_{\Gamma_\infty}y^3\left(\frac{y^2-1}{y^2-\eta^2}\right)^{1/2}dy=1, \tag{7.106}$$

which reduces by contour integration to

$$\hat{T}=\frac{t}{4}(t-\eta^2)(3\eta^2+1)=1. \tag{7.107}$$

However, as $t\uparrow t^{(1)}_{+,4}$, $T(\eta_1(t),\eta_2(t),t)\to\hat{T}(\hat{\eta}_2,\hat{t})$, so that we also have

$$\frac{\hat{t}}{4}(1-\hat{\eta}_2^2)(3\hat{\eta}_2^2+1)=1. \tag{7.108}$$

Also

$$A(\eta_1(t),\eta_2(t),t)\to\frac{2t}{\pi}\int_0^{\eta_2}\left(\frac{s^2-\hat{\eta}_2^2}{s^2-1}\right)^{1/2}s\left(s^2+\frac{\hat{\eta}_2^2-1}{2}\right)ds,$$

which must be zero. This requires that $(1-\hat{\eta}_2^2)/2<\hat{\eta}_2^2$, and so $\hat{\eta}_2^2>1/3$.

Now the parabola $(1-\eta^2)(3\eta^2+1)$ achieves its maximum also at $\eta^2=1/3$. It follows that for $t>\hat{t}$, we can, and do, chose $1>\eta=\eta(t)>\hat{\eta}_2$ to solve $\hat{T}(\eta(t),t)=1$.

The condition

$$\int_{-\eta}^{\eta}\left(H\psi+\frac{2t}{\pi}z^3\right)dz=\int_{-\eta}^{\eta}\frac{2t}{\pi}R(z)^{1/2}\,\hat{h}_4(z)\,dz=0$$

is automatic, as the integrand is odd. Also for

$$z \in J, \qquad \psi(z) = \frac{2it}{\pi} R_+^{1/2}(z)\, z \left(z^2 - \frac{1-\eta_2^2}{2}\right).$$

But as $\eta(t) > \hat{\eta}_2 > 1/\sqrt{3}$, we see that $z^2 - (1-\eta(t)^2)/2$ is positive on J, and hence $\psi(z) > 0$ on J.

Finally we show that

$$\begin{aligned}\int_{-\eta}^{\lambda} &\left(H\psi + \frac{2t}{\pi} z^3\right) dz \\ &= \frac{2t}{\pi} \int_{-\eta}^{\lambda} \left(\frac{z^2-\eta^2}{z^2-1}\right)^{1/2} z \left(z^2 + \frac{\eta^2-1}{2}\right) dz \leqslant 0 \\ &\text{for} \quad -\eta < \lambda < \eta. \end{aligned} \tag{7.109}$$

For $0 \leqslant \mu \leqslant 1$, set

$$G(\mu) = \int_0^{\mu} \left(\frac{\mu-u}{1-u}\right)^{1/2} \left(u + \frac{\mu-1}{2}\right) du.$$

Now

$$G'(\mu) = \frac{3\mu-1}{4} \int_0^{\mu} \left(\frac{\mu-u}{1-u}\right)^{1/2} \frac{du}{\mu-u},$$

which is positive for $1/3 < \mu < 1$ and negative for $0 < \mu < 1/3$. On the other hand $G(0) = 0$ and $G(1) > 0$. It follows that there exists a unique $\mu = \hat{\mu} \in (1/3, 1)$ such that $G(\hat{\mu}) = 0$.

Now $\int_{-\eta}^{0} (H\psi + (2t/\pi) z^3)\, dz = (-t/\pi)\, G(\eta(t)^2)$, and as $\eta(t)^2 > 1/3$, we must have $G(\eta(t)^2) > 0$ and hence $\int_{-\eta}^{0} (H\psi + (2t/\pi) z^3)\, dz < 0$. On the other hand, as $\eta^2 > 1/3$, we see from (7.109) that $\int_{-\eta}^{\lambda} (H\psi + (2t/\pi) z^3)\, dz < 0$ for λ near $-\eta$. If $\int_{-\eta}^{\lambda} (H\psi + (2t/\pi) z^3) dz > 0$ for some $-\eta < \lambda < 0$, it follows that the function must have at least *two* critical points in $(-\eta, 0)$. But clearly

$$\frac{d}{d\lambda} \int_{-\eta}^{\lambda} \left(H\psi + \frac{2t}{\pi} z^3\right) dz = \frac{2t}{\pi} R^{1/2} z \left(z^2 + \frac{\eta^2-1}{2}\right),$$

has only one critical point in this interval, which is a contradiction. Hence $\int_{-\eta}^{\lambda} (H\psi + (2t/\pi) z^3)\, dz < 0$ for $-\eta < \lambda \leqslant 0$; the same is true for $0 \leqslant \lambda < \eta$, as the function is even. This proves (7.109), and completes the proof of case **B** of Theorem 1.52.

It remains to prove cases $\mathbf{C}_1$ and $\mathbf{C}_2$ of Theorem 1.52. In the general case C_1, the proof involves modifications of the proof already given in the case $q=2$. The crucial fact is that for all $t > t^{(1)}_{+,2q}$, the analog of $\hat{h}_4$ (see (7.105)) is odd and has 3 roots in $(-1, 1)$. This can be proved in turn by using an analog of Lemma 7.11. On the other hand, the proof in the general case $\mathbf{C}_2$ involves modifications of the proof given below of case $\mathbf{B}_2$ in Theorem 1.60. Thus proofs of $\mathbf{C}_1$ and $\mathbf{C}_2$ involve ideas which are presented in other parts of this paper, and we present no further details.

8. PROOF OF THEOREM 1.60—THE CASE $V(x) = tx^{2q+1}$

We take $m = 2q+1$, and $V = tx^{2q+1}$. We will begin this section by considering the explicit case $q = 0$, and t large. We suppose that $J = (\eta, 1)$. From Theorem 1.38, we must have

$$\psi = Re\ F_+, \tag{8.1}$$

where

$$F(z) = \left(\frac{z-\eta}{z-1}\right)^{1/2} \frac{1}{\pi i} \int_\eta^1 \frac{-ti}{2\pi} \left(\frac{y-1}{y-\eta}\right)^{1/2} \frac{dy}{y-z} \tag{8.2}$$

$$= \frac{-ti}{2\pi}\left[1 - \left(\frac{z-\eta}{z-1}\right)^{1/2}\right]. \tag{8.3}$$

The condition $\int \psi = 1$ may be evaluated explicitly, and we find

$$\frac{t}{2}\frac{1-\eta}{2} = 1, \tag{8.4}$$

and we may solve for $\eta(t) \in [-1, 1]$ provided $t > 2$.

We will now verify that for $t > 2$, ψ defined by (8.1) satisfies the variational conditions (5.7)–(5.10). In the present case, we must only verify

$$\psi(x) > 0 \qquad \text{for} \quad y \in (\eta, 1), \tag{8.5}$$

$$-\int_\lambda^\eta H\psi + \frac{t}{2\pi}\, dy \leqslant 0 \qquad \text{for all} \quad \lambda \in [-1, \eta]. \tag{8.6}$$

As $F_+ = \psi + iH\psi$, we learn from (8.3) that

$$\psi(y) = \frac{it}{2\pi}\left(\frac{y-\eta}{y-1}\right)_+^{1/2}, \qquad y \in (\eta, 1), \tag{8.7}$$

$$H\psi + \frac{t}{2\pi} = \frac{t}{2\pi}\left(\frac{y-\eta}{y-1}\right)^{1/2}, \qquad y \in [-1, \eta], \tag{8.8}$$

and inequalities (8.5), (8.6) follow.

It is thus clear that we can solve for η and that the associated ψ is indeed the maximizer, provided $\eta \in [-1, 1]$. When $t = 2$, we must have $\eta = -1$, and for $t < 2$, we see that $\eta < -1$, which implies that ψ is not an admissible candidate. In particular, setting $t^{(1)}_{+,1} = 2$, we have proved case A_1 of Theorem 1.60.

To extend to the region $0 < t < 2$, we follow the analogy provided by the case m even, $V = -tx^m$. Here we suppose that $J = (-1, 1)$. From Theorem 1.38, we must have $\psi = Re\, F_+$, where now

$$F(z) = \frac{-ti}{2\pi}\left[1 - \left(\frac{z+1}{z-1}\right)^{1/2}\right] + \frac{i\gamma/\pi}{(z^2-1)^{1/2}}. \tag{8.9}$$

To satisfy $\int \psi = 1$, we must have

$$\gamma = 1 - \frac{t}{2}. \tag{8.10}$$

Clearly for all $t \in (0, 2)$, $\gamma > 0$, and hence

$$\psi(y) = \frac{it}{2\pi}\left(\frac{y+1}{y-1}\right)^{1/2} + \frac{i\gamma}{\pi}\frac{1}{(y^2-1)^{1/2}} \geqslant 0$$

for all $y \in J = (-1, 1)$. Thus we have produced the maximizer for all $t \in (0, \infty)$. This completes the proof of Case A of Theorem 1.60.

We now consider the case $q = 1$. Again we will begin with t large, and suppose that $J = (\eta, 1)$. From Theorem 1.38, we must have

$$\psi = Re\, F_+, \tag{8.11}$$

where now

$$F(z) = \left(\frac{z-\eta}{z-1}\right)^{1/2} \frac{1}{\pi i}\int_\eta^1 \frac{-3it}{2\pi}\, y^2 \left(\frac{y-1}{y-\eta}\right)^{1/2} \frac{dy}{y-z}, \tag{8.12}$$

or, in a more useful form,

$$F(z) + \frac{3it}{2\pi} z^2 = \frac{3it}{2\pi} \left(\frac{z-\eta}{z-1} \right)^{1/2} h(z), \tag{8.13}$$

where $h(z)$ is given by an integral over a now familiar, large, counterclockwise contour,

$$\begin{aligned} h(z) = h(z; \eta) &= \frac{1}{2\pi i} \oint_{\Gamma_\infty} y^2 \left(\frac{y-1}{y-\eta} \right)^{1/2} \frac{dy}{y-z} \\ &= z^2 - \frac{1-\eta}{2} z - \frac{1-\eta}{8} (3\eta + 1). \end{aligned} \tag{8.14}$$

As $F_\pm = \pm\psi + iH\psi$, we see that $\int \psi = \frac{1}{2} \int_\eta^1 F_+ - F_- \, dx = (-1/2) \oint_{\Gamma_\infty} F(z)\, dz$, where the contour Γ_∞ is the same as above. But then $\int \psi = -i\pi c$, where $F(z) = c/z + O(1/z^2)$ as $z \to \infty$. Using (8.12), the condition $\int \psi = 1$ now becomes

$$T(\eta) = 1, \tag{8.15}$$

where T is given by

$$T(\eta) = \frac{-3it}{2\pi} \int_\eta^1 y^2 \left(\frac{y-1}{y-\eta} \right)^{1/2} dy. \tag{8.16}$$

We also have the following useful representation for T:

$$T = \frac{-3t}{2} \frac{1}{2\pi i} \oint_{\Gamma_\infty} y^2 \left(\frac{y-1}{y-\eta} \right)^{1/2} dy = \frac{3t}{32} (1-\eta)(1 + 2\eta + 5\eta^2). \tag{8.17}$$

From (8.17) we obtain the useful formula

$$\begin{aligned} \frac{\partial}{\partial \eta} T &= \frac{-3t}{2} \frac{1}{2\pi i} \oint_{\Gamma_\infty} y^2 \left(\frac{y-1}{y-\eta} \right)^{1/2} \frac{1/2}{y-\eta} dy \\ &= \frac{-3t}{4} h(\eta). \end{aligned} \tag{8.18}$$

From the polynomial representation of T in (8.17), we conclude that for given $\delta > 0$, there exists $\tilde{t}(\delta) < \infty$ such that for all $t > \tilde{t}$, there exists a (unique) solution $\eta(t)$ to the equation (8.15) with $0 < 1 - \eta(t) < \delta$.

We will now verify that for t sufficiently big, ψ defined by (8.11) satisfies the variational conditions (5.7)–(5.10). As in the case $q=0$ above, we must only verify

$$\psi(y)>0 \qquad \text{for all} \quad y\in J, \tag{8.19}$$

$$-\int_{\lambda}^{\eta} H\psi+\frac{3t}{2\pi}\,y^2\leqslant 0 \qquad \text{for all} \quad \lambda\in[-1,\eta]. \tag{8.20}$$

As $F_+=\psi+iH\psi$, we have

$$\psi(y)=\frac{3it}{2\pi}\left(\frac{y-\eta}{y-1}\right)_+^{1/2} h(y), \qquad \text{for} \quad y\in(\eta,1), \tag{8.21}$$

$$H\psi(y)+\frac{3t}{2\pi}\,y^2=\frac{3t}{2\pi}\left(\frac{y-\eta}{y-1}\right)^{1/2} h(y), \qquad \text{for} \quad y\in(-1,\eta). \tag{8.22}$$

Firstly, observe that for δ sufficiently small, and $0<1-\eta<\delta<\frac{1}{2}$, $h(z)$ defined by (8.14) satisfies

$$h(z)>0 \qquad \text{for} \quad z\in(\tfrac{1}{2},1), \tag{8.23}$$

and hence for $t>\tilde{t}(\delta)$, (8.19) holds, and by (8.22), $H\psi+(3t/2\pi)\,y^2>0$ for $y\in(\frac{1}{2},\eta)$, which implies that

$$-\int_{\lambda}^{\eta}\left(H\psi+\frac{3t}{2\pi}\,y^2\right)dy<0 \qquad \text{for} \quad \lambda\in\left(\frac{1}{2},\eta\right). \tag{8.24}$$

Secondly, note that if we let $\eta\to 1$, then $H\psi+(3t/2\pi)\,y^2\to(3t/2\pi)\,y^2$, and hence for each $\lambda\in$ (-1,1),

$$\lim_{\eta\to 1} -\int_{\lambda}^{\eta} H\psi+\frac{3t}{2\pi}\,y^2\,dy=-\int_{\lambda}^{1}\frac{3t}{2\pi}\,y^2\,dy=\frac{t(\lambda^3-1)}{2\pi}<0. \tag{8.25}$$

Thus, by continuity, $-\int_{\lambda}^{\eta} H\psi+(3t/2\pi)\,y^2\,dy<0$ for all $\lambda\leqslant\frac{1}{2}$, say, and for $0<1-\eta<\delta'$, $\delta'>0$ small. Taking $\bar{\delta}=\min(\delta,\delta')$, we conclude that for $t>\tilde{t}(\bar{\delta})$, $-\int_{\lambda}^{\eta} H\psi+(3t/2\pi)\,y^2\,dy<0$ for all $\lambda\in[-1,\eta)$, as desired. Thus for $t>\tilde{t}(\bar{\delta})$, ψ defined by (8.11) is the maximizer.

For future reference, a plot of the function $(1/t)\,T(\eta)$ versus η is shown in Fig. 9. Note in particular that this cubic is monotone decreasing on $((3+2\sqrt{6})/15,1)$, with a local maximum of $(9+\sqrt{6})/75$ at $(3+2\sqrt{6})/15$. Define $\underline{\eta}=(3+2\sqrt{6})/15$, and $\underline{t}=75/(9+\sqrt{6})$. Thus for $t\in(\underline{t},\infty)$, we can certainly solve for η uniquely in $(\underline{\eta},1)$ so that $T(\eta)=1$.

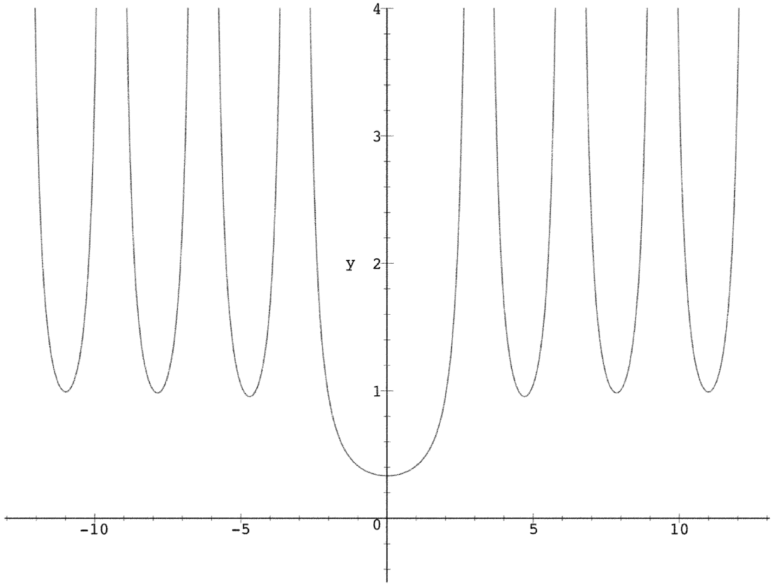

FIG. 4.1. $f(x)$ on $[-4\pi, 4\pi]$.

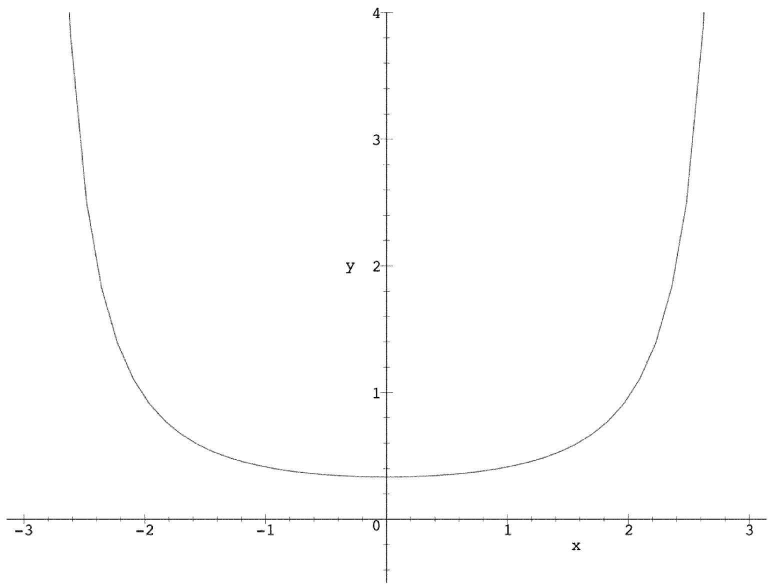

FIG. 4.2. $f(x)$ on $[-\pi, \pi]$.

Now we will show that at $\hat{t}$, the inequality (8.20) remains true, but with equality at some $\bar{\lambda} \in (-1, \eta)$. In other words, $\hat{t}$ is determined by the pair of equations

$$\int_{\lambda}^{\eta} H\psi(y) + \frac{3t}{2\pi} y^2 \, dy = 0 \tag{8.30}$$

$$H\psi(\lambda) + \frac{3t}{2\pi} \lambda^2 = 0, \tag{8.31}$$

for some $\lambda = \bar{\lambda} \in (-1, \eta)$, together with $\int \psi \, dx = 1$. It is *important* to observe from (8.22) that the variable t actually drops out of equations (8.30), (8.31). Hence we may solve (8.30), (8.31) to determine $\eta = \hat{\eta}$, in particular, independent of t; then we choose $t = \hat{t}$ to solve $\int \psi \, dx = 1$, where ψ is determined by $(\hat{\eta}, 1)$.

Recall that for η near 1, $h(z)$ possesses two roots $z_{\pm}$, $z_- < 0 < z_+$, both in a vicinity of the origin,

$$z_{\pm} = \frac{1-\eta}{4} \pm \frac{1}{4} \sqrt{(1-\eta)(5\eta+3)}. \tag{8.32}$$

Define

$$I(\eta) = \int_{z_-(\eta)}^{\eta} H\psi + \frac{3t}{2\pi} y^2 \, dy. \tag{8.33}$$

As (8.20) holds for η near 1 with strict inequality in the interval $[-1, \eta)$, we know that $I(\eta) > 0$. Differentiating, we find using (8.22) that

$$\begin{aligned} \frac{\partial}{\partial \eta} I &= \int_{z_-}^{\eta} \frac{\partial}{\partial \eta} H\psi(y) \, dy \\ &= \frac{T_\eta}{\pi} \int_{z_-}^{\eta} \frac{1}{((y-\eta)(y-1))^{1/2}} \, dy \end{aligned} \tag{8.34}$$

where the second line in (8.34) follows from the formula

$$\frac{\partial}{\partial \eta} F(z) = \frac{i}{\pi} \left(\frac{z-\eta}{z-1} \right)^{1/2} \frac{T_\eta}{z-\eta}, \tag{8.35}$$

which will be proven below.

Now for $\eta > \underline{\eta}$, $T_\eta < 0$, and hence $I_\eta > 0$. Also, as $z_+(\underline{\eta}) = \underline{\eta}$,

$$I(\underline{\eta}) = \int_{z_-(\underline{\eta})}^{z_+(\underline{\eta})} \frac{3t}{2\pi} \left(\frac{y-\underline{\eta}}{y-1} \right)^{1/2} h(y) \, dy < 0.$$

Thus there is a unique η^* in $(\underline{\eta}, 1)$ such that

$$I(\eta^*) = 0, \tag{8.36}$$

$$I(\eta) > 0 \qquad \text{for all} \quad \eta \in (\eta^*, 1), \tag{8.37}$$

$$I(\eta) < 0 \qquad \text{for all} \quad \eta \in (\underline{\eta}, \eta^*). \tag{8.38}$$

The above calculations show in particular that $\eta = \eta^*$, $\lambda = z_-(\eta^*)$ solve (8.30) and (8.31). By the observation after (8.31), note that η^* is independent of t.

Now observe that for $\eta > \eta^*$, $h(z) > 0$ for $z \in [\eta, 1]$; indeed, for $\eta > \eta^*$, $I(\eta) = \int_{z_-(\eta)}^{\eta} (3t/2\pi)((y-\eta)/(y-1))^{1/2} h(y)\, dy > 0$ and hence the second root $z_+(\eta)$ of $h(z)$ must be less than η. Hence $\psi(x) = (3it/2\pi)((x-\eta)/(x-1))^{1/2} h(x) > 0$ for $x \in (\eta, 1)$. Further,

$$-\int_{\lambda}^{\eta} \left(H\psi + \frac{3t}{2\pi} y^2 \right) dy = -\left[I(\eta) + \int_{\lambda}^{z_-(\eta)} \left(H\psi + \frac{3t}{2\pi} y^2 \right) dy \right].$$

Noting that $z_-(\eta) < z_+(\eta) < \eta$, and considering the three cases $\lambda \in (-1, z_-(\eta))$, $\lambda \in (z_-(\eta), z_+(\eta))$, $\lambda \in (z_+(\eta), \eta)$ separately, we conclude that

$$-\int_{\lambda}^{\eta} \left(H\psi + \frac{3t}{2\pi} y^2 \right) dy \leqslant 0 \qquad \text{for} \quad \lambda \in [-1, \eta). \tag{8.39}$$

Thus for $\eta \geqslant \eta^*$, ψ defined by (8.11) satisfies the variational conditions (8.19), (8.20). As $\eta^* > \underline{\eta}$, we may also chose $t = t^*$ such that (8.15) is satisfied for $\eta = \eta^*$. For $t > t^*$, we have $\eta(t) > \eta^*$ and hence the corresponding ψ is the maximizer. On the other hand, if t is slightly less than t^*, then $\eta(t) < \eta^*$, and hence $I(\eta(t)) < 0$, which shows that the condition (8.20) fails. We conclude that

$$\hat{t} = t^*, \qquad \hat{\eta} = \eta^*. \tag{8.40}$$

The proof that ψ constructed above is the maximizer for $t > \hat{t}$ will be complete when we establish formula (8.35). To accomplish this, we begin with the following representation for the function $F(z)$,

$$F(z) = \frac{-1}{z\pi i} \left(\frac{z-\eta}{z-1} \right)^{1/2} \int_J \frac{-3it}{2\pi} y^2 \left(\frac{y-1}{y-\eta} \right)^{1/2} \left(1 + \left(\frac{y}{z}\right) + \left(\frac{y}{z}\right)^2 + \cdots \right) dy$$

$$= \left(\frac{z-\eta}{z-1} \right)^{1/2} \frac{-1}{z\pi i} \sum_{j=}^{\infty} Y_j z^{-j}, \tag{8.41}$$

where Y_j is defined by

$$Y_j = \int_J \frac{-3it}{2\pi}\, y^{2+j} \left(\frac{y-1}{y-\eta}\right)^{1/2} dy. \tag{8.42}$$

Note that $Y_0 = T$.

We may differentiate (8.41) with respect to η, and we find

$$\begin{aligned} F_\eta &= \frac{-1}{z\pi i}\left(\frac{z-\eta}{z-1}\right)^{1/2} \sum_{j=0}^{\infty} \left(\frac{-1/2}{z-\eta}\, Y_j + Y_{j,\eta}\right) z^{-j} \\ &= \frac{-1}{z\pi i}\left(\frac{z-\eta}{z-1}\right)^{1/2} \frac{-1/2}{z-\eta}\left[-2z Y_{0,\eta} + \sum_{j=0}^{\infty} z^{-j}(Y_j - 2(Y_{j+1,\eta} - n Y_{j,\eta}))\right]. \end{aligned} \tag{8.43}$$

However, we know that

$$Y_{j+1} - \eta Y_j = \int_J \frac{-3it}{2\pi}\, y^{2+j}\left(\frac{y-1}{y-\eta}\right)^{1/2} (y-\eta)\, dy, \tag{8.44}$$

and hence, differentiating, we obtain

$$Y_{j+1,\eta} - \eta Y_{j,\eta} - Y_j = \frac{-1}{2}\int_J \frac{-3it}{2\pi}\, y^{2+j}\left(\frac{y-1}{y-\eta}\right)^{1/2} dy, \tag{8.45}$$

which implies

$$2(Y_{j+1,\eta} - \eta Y_{j,\eta}) = Y_j. \tag{8.46}$$

Inserting (8.46) into (8.43), we have established

$$F_\eta = \frac{i}{\pi}\left(\frac{z-\eta}{z-1}\right)^{1/2} \frac{Y_{0,\eta}}{z-\eta}, \tag{8.47}$$

as advertised.

Setting $t^{(1)}_{+,3} = \hat{t}$, we have completed the proof of case B_1 of Theorem 1.60.

Summarizing, for $t^{(1)}_{+,3} < t$, we have constructed the solution of the maximization problem, namely $\psi = Re\, F_+$, with F defined by (8.12). For $t < t^{(1)}_{+,3}$, we must add an interval to the support J. We will suppose that $J = (\eta_1, \eta_2) \cup (\eta_3, 1)$, $-1 < \eta_1 < \eta_2 < \eta_3 < 1$. Under this assumption, we must have (from Theorem 1.38) that $\psi = Re\, F_+$, where now $F(z)$ is given by

$$F(z) = \frac{-3it}{2\pi}\frac{R^{1/2}}{\pi i}\int_J \frac{y^2}{R(y)^{1/2}}\frac{dy}{y-z}, \tag{8.48}$$

where

$$R(z) = \frac{(z-\eta_1)(z-\eta_2)(z-\eta_3)}{z-1}. \tag{8.49}$$

The following representation will again prove useful,

$$F + \frac{3it}{2\pi} z^2 = \frac{3it}{2\pi} R(z)^{1/2} h_1(z), \tag{8.50}$$

where now

$$\begin{aligned} h_1(z) &= \frac{1}{2\pi i} \oint_{\Gamma_\infty} \frac{y^2}{R(y)^{1/2}} \frac{dy}{y-z} \\ &= z + \frac{\eta_1 + \eta_2 + \eta_3 - 1}{2}. \end{aligned} \tag{8.51}$$

In contrast to the case of m even, we now have three η's to be determined. The condition that $F \in H^p$ implies that we must have

$$\frac{-3it}{2\pi} \int_J \frac{y^2}{R(y)^{1/2}} dy = 0. \tag{8.52}$$

Next, we impose the condition $\int \psi = 1$, which yields the equation

$$\frac{-3it}{2\pi} \int_J \frac{y^3}{R(y)^{1/2}} dy = 1. \tag{8.53}$$

The third condition, that $L\psi(\eta_2) + t\eta_2^3 = L\psi(\eta_3) + t\eta_3^3$, implies that we must have

$$\int_{\eta_2}^{\eta_3} H\psi + \frac{3t}{2\pi} y^2 \, dy = 0. \tag{8.54}$$

Define T_0, T_1, and A_1 by

$$T_0 = \frac{-3it}{2\pi} \int_J \frac{y^2}{R(y)^{1/2}} dy, \tag{8.55}$$

$$T_1 = \frac{-3it}{2\pi} \int_J \frac{y^3}{R(y)^{1/2}} dy, \tag{8.56}$$

$$A_1 = \int_{\eta_2}^{\eta_3} H\psi + \frac{3t}{2\pi} y^2 \, dy. \tag{8.57}$$

These functions are defined a priori for $-1<\eta_1<\eta_2<\eta_3<1$, however familiar computations show that

$$T_j \frac{3it}{4\pi} \oint_{\Gamma_\infty} \frac{y^{2+j}}{R(y)^{1/2}}\, dy, \qquad j=0, 1, \tag{8.58}$$

where again Γ_∞ is a counterclockwise contour of large radius. But $R(y)=((y-\eta_1)(y-\eta_2)(y-\eta_3))/(y-1)$, and it is clear that $T_j=T_j(\eta_1, \eta_2, \eta_3)$ has a real analytic continuation to $-1<\eta_1, \eta_2, \eta_3<1$, in particular. On the other hand, from (8.50), (8.51),

$$\begin{aligned} A_1 &= A_1(\eta_1, \eta_2, \eta_3) \\ &= \int_{\eta_2}^{\eta_3} \frac{3t}{2\pi} \left(\frac{(y-\eta_1)(y-\eta_2)(y-\eta_3)}{y-1} \right)^{1/2} \left(y+\frac{\eta_1+\eta_2+\eta_3-1}{2} \right) dy, \end{aligned} \tag{8.59}$$

and direct differentiation shows that A_1 and its first partial derivatives have continuous extensions to $-1<\eta_1\leqslant\eta_2<\eta_3<1$. Moreover, it is easy to see that for $-1<\eta<\eta_3<1$,

$$\lim_{\substack{\eta_1, \eta_2 \to \eta \\ \eta_1<\eta_2}} \frac{\partial A(\eta_1, \eta_2, \eta_3)}{\partial \eta_1} = \lim_{\substack{\eta_1, \eta_2 \to \eta \\ \eta_1<\eta_2}} \frac{\partial A(\eta_1, \eta_2, \eta_3)}{\partial \eta_2}. \tag{8.60}$$

Simple calculus now shows that if we set

$$A_1(\eta_1, \eta_2, \eta_3) \equiv A_1(\eta_2, \eta_1, \eta_3) \qquad \text{for} \quad -1<\eta_2<\eta_1<\eta_3<1, \tag{8.61}$$

then A_1 extends to a C^1 function on $-1<\eta_1, \eta_2<\eta_3<1$. In summary we see that T_0, T_1, and A_1 may be viewed as C^1 functions on the region $-1<\eta_1, \eta_2<\eta_3<1$.

The three conditions (8.52)–(8.54), can be written

$$T_0=0 \tag{8.62}$$

$$T_1=1 \tag{8.63}$$

$$A_1=0. \tag{8.64}$$

We view these equations on the extended region $-1<\eta_1, \eta_2<\eta_3<1$.

Claim. *The triple*

$$(\eta_1, \eta_2, \eta_3)=(z_-(\eta^*), z_-(\eta^*), \eta^*)\in \{(\eta_1, \eta_2, \eta_3)\colon -1<\eta_1, \eta_2<\eta_3<1\}$$

solves (8.62), (8.63), *and* (8.64) *for* $t=t^{(1)}_{+,3}$.

Indeed,

$$T_0(z_-(\eta^*), z_-(\eta^*), \eta^*) = \frac{3it}{4\pi} \oint_{\Gamma_\infty} y^2 \left(\frac{y-1}{y-\eta^*}\right)^{1/2} \frac{dy}{y - z_-(\eta^*)}$$
$$= \frac{-3t}{2} h(z_-(\eta^*), \eta^*) = 0 \qquad (8.65)$$

by (8.14), and the definition of $z_\pm$. Also,

$$T_1(z_-(\eta^*), z_-(\eta^*), \eta^*) = \frac{3it}{4\pi} \oint_{\Gamma_\infty} y^3 \left(\frac{y-1}{y-\eta^*}\right)^{1/2} \frac{dy}{y - z_-(\eta^*)}$$
$$= \frac{3it}{4\pi} \oint_{\Gamma_\infty} y^2 (y - z_-(\eta^*)) \left(\frac{y-1}{y-\eta^*}\right)^{1/2} \frac{dy}{y - z_-(\eta^*)},$$
$$\text{as} \quad T_0 = 0,$$
$$= T(\eta^*) = 1 \qquad (8.66)$$

by (8.17) and (8.15). Finally, from (8.59),

$$A_1(z_-(\eta^*), z_-(\eta^*), \eta^*)$$
$$= \frac{3t}{2\pi} \int_{z_-(\eta^*)}^{\eta^*} \left(\frac{y-\eta^*}{y-1}\right)^{1/2} (y - z_-(\eta^*)) \left(y + \frac{2z_-(\eta^*) + \eta^* - 1}{2}\right) dy. \qquad (8.67)$$

But $z_-(\eta^*) + ((1-\eta^* - 2z_-(\eta^*))/2) = (1-\eta^*)/2$, and hence $(y - z_-(\eta^*))(y + (2z_-(\eta^*) + \eta^* - 1)/2) = h(y; \eta^*)$ (see (8.14)). Thus $A_1(z_-(\eta^*), z_-(\eta^*), \eta^*) = 0$ by (8.22) and (8.36). This establishes the Claim.

Thus we have a solution of the system (8.62-8.64) for $t = t^{(1)}_{+,3}$, and the task now is to show that this system of equations can be solved for $t < t^{(1)}_{+,3}$, for $\{\eta_j\}_{j=1}^3$ in a vicinity of the solution $(z_-(\eta^*), z_-(\eta^*), \eta^*)$. In particular, we will show that there exists a solution (η_1, η_2, η_3) to the system for all $t \in (25/6, t^{(1)}_{+,3})$. For $t < 25/6$, it will then be shown that J once again reduces to a single interval $(\eta, 1)$, $-1 < \eta < 1$.

We first establish that there is a solution to the system (8.62)–(8.64) for $t^{(1)}_{+,3} - \varepsilon < t < t^{(1)}_{+,3}$, for some small $\varepsilon > 0$. This will be accomplished by showing that we can choose η_1 and η_3 to solve (8.62) and (8.64), as functions of η_2, for η_2 near $z_-(\eta^*)$. Then, evaluating T_1 at $(\eta_1(\eta_2), \eta_2, \eta_3(\eta_2))$, we will show that we can chose η_2 to solve (8.63), and hence the system.

To prove that we can chose (η_1, η_3) to solve (8.62) and (8.64), we need only compute the Jacobian of the transformation $(\eta_1, \eta_3) \mapsto (T_0, A_1)$, and evaluate at the solution $(z_-(\eta^*), z_-(\eta^*), \eta^*)$. A computation entirely similar to (8.34) shows that for $j=1, 2, 3$,

$$\frac{\partial A_1}{\partial \eta_j} = \frac{T_{0,\eta_j}}{\pi} \int_{\eta_2}^{\eta_3} \frac{R^{1/2}}{z-\eta_j}\, dz, \tag{8.68}$$

where we use the formula

$$F_{\eta_j} = \frac{i}{\pi} \frac{R^{1/2}}{z-\eta_j} T_{0,\eta_j}, \tag{8.69}$$

which can be derived in turn in the same way as formula (8.35). Thus we have

$$\begin{aligned} \begin{vmatrix} T_{0,\eta_1} & T_{0,\eta_3} \\ A_{1,\eta_1} & A_{1,\eta_3} \end{vmatrix} &= \frac{T_{0,\eta_1} T_{0,\eta_3}}{\pi} \begin{vmatrix} 1 & 1 \\ \int_{\eta_2}^{\eta_3} \frac{R^{1/2}}{z-\eta_1} & \int_{\eta_2}^{\eta_3} \frac{R^{1/2}}{z-\eta_3}\, dz \end{vmatrix} \\ &= \frac{T_{0,\eta_1} T_{0,\eta_3}}{\pi} \int_{\eta_2}^{\eta_3} \frac{\eta_3-\eta_1}{(z-\eta_1)(z-\eta_3)}\, dz. \end{aligned} \tag{8.70}$$

Differentiating (8.58) with respect to η_j, we obtain

$$\begin{aligned} T_{0,\eta_j} &= \frac{3it}{4\pi} \oint_{\Gamma_\infty} \frac{y^2}{R(y)^{1/2}} \frac{1/2}{y-\eta_j}\, dy \\ &= \frac{-3t}{4} \left(\eta_j + \frac{\eta_1+\eta_2+\eta_3-1}{2} \right) \\ &= \frac{-3t}{4} h_1(\eta_j). \end{aligned} \tag{8.71}$$

However, from (8.59) and the condition $A_1(z_-(\eta^*), z_-(\eta^*), \eta^*)=0$, we see that $h_1(\eta^*)>0$ and $h_1(z_-(\eta^*))<0$. Hence

$$T_{0,\eta_1}(z_-, z_-, \eta^*) > 0, \tag{8.72}$$

$$T_{0,\eta_3}(z_-, z_-, \eta^*) < 0. \tag{8.73}$$

It follows that the Jacobian (8.70) is positive.

We now use the implicit function theorem. Indeed, since the Jacobian (8.70) is nonzero, we may solve for (η_1, η_3) as C^1 functions of η_2 in a vicinity of $(z_-(\eta^*), \eta^*)$, so that $(\eta_1(\eta_2), \eta_3(\eta_2)) \to (z_-(\eta^*), \eta^*)$ as $\eta_2 \to z_-(\eta^*)$.

Furthermore as we will see below, for $z_{-}(\eta^{*})<\eta_2<z_{-}(\eta^{*})+\varepsilon_1$, ε_1 small, we have

$$\eta_1(\eta_2)<z_{-}(\eta^{*})<\eta_2 \qquad \text{and} \qquad \eta_2<\eta_3(\eta_2)<\eta^{*}. \tag{8.74}$$

Take $t=t^{(1)}_{+,3}$, and define

$$G(\eta_2)=T_1(\eta_1(\eta_2),\eta_2,\eta_2(\eta_2)). \tag{8.75}$$

We thus have $G(z_{-}(\eta^{*}))=1$, and we shall prove that there is a unique solution η_2 (near $z_{-}(\eta^{*})$) of the equation $G(\eta_2)=1$, for $0<t^{(1)}_{+,3}-\varepsilon<t<t^{(1)}_{+,3}$.

Following the same general argument used to obtain (8.46), we have

$$T_{1,\eta_j}-\eta_j T_{0,\eta_j}=T_0, \tag{8.76}$$

and evaluating at a solution of $T_0(\eta_1,\eta_2,\eta_3)=0$, we obtain

$$T_{1,\eta_j}=\eta_j T_{0,\eta_j}. \tag{8.77}$$

Thus, differentiating G with respect to η_2, we have

$$\begin{aligned} G_{\eta_2}&=T_{1,\eta_1}\frac{\partial\eta_1}{\partial\eta_2}+T_{1,\eta_2}+T_{1,\eta_3}\frac{\partial\eta_3}{\partial\eta_2}\\ &=\eta_1 T_{0,\eta_1}\frac{\partial\eta_1}{\partial\eta_2}+\eta_2 T_{0,\eta_2}+\eta_3 T_{0,\eta_3}\frac{\partial\eta_3}{\partial\eta_2}. \end{aligned} \tag{8.78}$$

Using the pair of equations

$$T_0(\eta_1,\eta_2,\eta_3)=0, \tag{8.79}$$

$$A_1(\eta_1,\eta_2,\eta_3)=0, \tag{8.80}$$

together with the derivative relation (8.68) we may obtain

$$T_{0,\eta_1}\frac{\partial\eta_1}{\partial\eta_2}=-T_{0,\eta_2}\frac{\begin{vmatrix} 1 & 1\\ \int\frac{R^{1/2}}{z-\eta_2} & \int\frac{R^{1/2}}{z-\eta_3}\end{vmatrix}}{\begin{vmatrix} 1 & 1\\ \int\frac{R^{1/2}}{z-\eta_1} & \int\frac{R^{1/2}}{z-\eta_2}\end{vmatrix}}. \tag{8.81}$$

$$T_{0,\eta_3}\frac{\partial\eta_3}{\partial\eta_2}=-T_{0,\eta_2}\frac{\begin{vmatrix}1 & 1\\ \int\frac{R^{1/2}}{z-\eta_1} & \int\frac{R^{1/2}}{z-\eta_3}\end{vmatrix}}{\begin{vmatrix}1 & 1\\ \int\frac{R^{1/2}}{z-\eta_1} & \int\frac{R^{1/2}}{z-\eta_3}\end{vmatrix}}. \tag{8.82}$$

As $T_{0,\eta_2}(z_-(\eta^*), z_-(\eta^*), \eta^*) = T_{0,\eta_1}(z_-(\eta^*), z_-(\eta^*), \eta^*) > 0$ by (8.72), we see that $\eta_1'(\eta_2) < 0$ for η_2 near $z_-(\eta^*)$: thus $\eta_1(\eta_2) < z_-(\eta^*)$ for $\eta_2 > z_-(\eta^*)$. But then from (8.73) and (8.82) we easily see that $\eta_3 < \eta^*$. This verifies (8.74).

Inserting formulae (8.81), (8.82) into (8.78), we obtain

$$G_{\eta_2}=\frac{T_{0,\eta_2}}{\begin{vmatrix}1 & 1\\ \int\frac{R^{1/2}}{z-\eta_1} & \int\frac{R^{1/2}}{z-\eta_3}\end{vmatrix}}\begin{vmatrix}1 & 1 & 1\\ \eta_1 & \eta_2 & \eta_3\\ \int\frac{R^{1/2}}{z-\eta_1} & \int\frac{R^{1/2}}{z-\eta_2} & \int\frac{R^{1/2}}{z-\eta_3}\end{vmatrix} \tag{8.83}$$

$$=\frac{T_{0,\eta_2}}{\int_{\eta_2}^{\eta_3} R^{1/2}\frac{1}{(z-\eta_3)(z-\eta_1)}\,dz}\int_{\eta_2}^{\eta_3}\frac{(\eta_3-\eta_2)(\eta_2-\eta_1)}{\left(\prod_{j=1}^{3}(z-\eta_j)(z-1)\right)^{1/2}}\,dz, \tag{8.84}$$

and since $T_{0,\eta_2}(z_-, z_-, \eta^*) = T_{0,\eta_1}(z_-, z_-, \eta^*) > 0$ by (8.72), we have established that $G_{\eta_2}(\eta_2) > 0$ for $z_-(\eta^*) < \eta_2 < z_-(\eta^*) + \varepsilon_1$.

However, as G is linear in t, the equation $G(\eta_2, t) = 1$ becomes $G(\eta_2, 1) = 1/t$. Now $G(z_-(\eta^*), 1) = 1/t^{(1)}_{+,3}$. Thus for some small $\varepsilon > 0$ we may chose $\eta_2 = \eta_2(t) \in (z_-(\eta^*), z_-(\eta^*) + \varepsilon_1)$ (uniquely) such that $G(\eta_2, t) = 1$ for $t \in (t^{(1)}_{+,3} - \varepsilon, t^{(1)}_{+,3})$. But then by (8.74),

$$\eta_1(t) < z_-(\eta^*) < \eta_2(t) < \eta_3(t) < \eta^*. \tag{8.85}$$

Thus we have established the existence of a solution $-1 < \eta_1(t) < \eta_2(t) < \eta_3(t) < 1$ to the system of equations (8.62)–(8.64), for $t^{(1)}_{+,3} - \varepsilon < t < t^{(1)}_{+,3}$.

Our next task is to show that the function F defined by (8.48), with $\eta_j = \eta_j(t)$, $1 \leqslant j \leqslant 3$, as above, satisfies the variational conditions, i.e., that the associated $\pi = Re\, F_+$ is indeed the maximizer.

Observe from (8.59) that the condition $A_1 = 0$ implies that the single zero of the function $h_1(z)$ defined by (8.51) must lie in the interval (η_2, η_3). Now as $F_\pm = \pm\psi + iH\psi$, we have from (8.50),

$$\psi(y) = \frac{3it}{2\pi} R(y)^{1/2} h_1(y) \qquad \text{for} \quad y \in J, \tag{8.86}$$

$$H\psi + \frac{3t}{2\pi} y^2 = \frac{3t}{2\pi} R(y)^{1/2} h_1(y) \qquad \text{for} \quad y \in \mathbf{R} \setminus J. \tag{8.87}$$

Thus, by familiar arguments, we conclude (using the fact that the (single) root of $h_1(z)$ lies in (η_2, η_3)) that

$$\psi(y) \geqslant 0 \qquad \text{for} \quad y \in J, \tag{8.88}$$

$$\int_{\eta_2}^{y} H\psi + \frac{3t}{2\pi} y^2 \, dy < 0 \qquad \text{for} \quad y \in (\eta_2, \eta_3), \tag{8.89}$$

$$\int_{y}^{\eta_1} H\psi + \frac{3t}{2\pi} y^2 \, dy > 0 \qquad \text{for} \quad y \in (-1, \eta_1), \tag{8.90}$$

and hence $\psi = Re\, F_+$ with F defined by (8.48) is the maximizer for $t^{(1)}_{+,3} - \varepsilon < t < t^{(1)}_{+,3}$.

Remark 8.91. For later reference, notice that (8.88)–(8.90) hold whenever $-1 < \eta_1 < \eta_2 < \eta_3 < 1$ solve (8.62)–(8.64).

Furthermore, as

$$h_1(\eta_1) < 0 \tag{8.92}$$

$$h_1(\eta_2) < 0 \tag{8.93}$$

$$h_1(\eta_3) > 0, \tag{8.94}$$

it follows from the third line of (8.71) that on the solution of (8.62)–(8.64),

$$T_{0,\eta_1} > 0 \tag{8.95}$$

$$T_{0,\eta_2} > 0 \tag{8.96}$$

$$T_{0,\eta_3} < 0. \tag{8.97}$$

For $t^{(1)}_{+,3} - \varepsilon < t < t^{(1)}_{+,3}$, we may evaluate the Jacobian of the transformation $(\eta_1, \eta_2, \eta_3) \mapsto (T_0, T_1, A_1)$, using calculations entirely similar to those used to evaluate the Jacobian in (8.70). Combining (8.68) and (8.77), we obtain

$$\begin{vmatrix} T_{0,\eta_1} & T_{0,\eta_2} & T_{0,\eta_3} \\ T_{1,\eta_1} & T_{1,\eta_2} & T_{1,\eta_3} \\ A_{1,\eta_1} & A_{1,\eta_2} & A_{1,\eta_3} \end{vmatrix} \tag{8.98}$$

$$= \frac{\prod_{j=1}^{3} T_{0,\eta_j}}{\pi} \int_{\eta_2}^{\eta_3} \frac{(\eta_{32} - \eta_1)(\eta_3 - \eta_2)(\eta_2 - \eta_1)}{\left(\prod_{j=1}^{3} (z - \eta_j)(z - 1)\right)^{1/2}} \, dz. \tag{8.99}$$

Since we know that $T_{0,\eta_j} \neq 0$, it follows that the Jacobian in (8.98) is nonzero for $-1 < \eta_1 < \eta_2 < \eta_3 < 1$ chosen to solve the system (8.62)–(8.64).

For $t_{+,3}^{(1)} - \varepsilon < t < t_{+,3}^{(1)}$, we may compute $T_{0,\eta_j}(\partial \eta_j / \partial t)$, using (8.68) and (8.77), by differentiating $T_0 = 0$, $T_1 = 1$, $A_1 = 0$. We have

$$T_{0,\eta_1} \eta_{1,t} = \frac{\dfrac{1}{t} \displaystyle\int_{\eta_2}^{\eta_3} \frac{R^{1/2}\, dz}{(z-\eta_2)(z-\eta_3)}}{\displaystyle\int_{\eta_2}^{\eta_3} \frac{(\eta_3 - \eta_1)(\eta_2 - \eta_1)\, dz}{((z-1)\prod_{j=1}^{3}(z-\eta_j))^{1/2}}} > 0, \tag{8.100}$$

$$T_{0,\eta_2} \eta_{2,t} = \frac{\dfrac{-1}{t} \displaystyle\int_{\eta_2}^{\eta_3} \frac{R^{1/2}\, dz}{(z-\eta_1)(z-\eta_3)}}{\displaystyle\int_{\eta_2}^{\eta_3} \frac{(\eta_3 - \eta_2)(\eta_2 - \eta_1)\, dz}{((z-1)\prod_{j=1}^{3}(z-\eta_j))^{1/2}}} < 0, \tag{8.101}$$

$$T_{0,\eta_3} \eta_{3,t} = \frac{\dfrac{1}{t} \displaystyle\int_{\eta_2}^{\eta_3} \frac{R^{1/2}\, dz}{(z-\eta_1)(z-\eta_2)}}{\displaystyle\int_{\eta_2}^{\eta_3} \frac{(\eta_3 - \eta_2)(\eta_3 - \eta_1)\, dz}{((z-1)\prod_{j=1}^{3}(z-\eta_j))^{1/2}}} < 0. \tag{8.102}$$

Remark. Recalling that $T_0(\eta_1, \eta_2, \eta_3, t)$ is linear in t, we observe that with respect to the natural time variable $\tilde{t} = 1/t$, (8.100), (8.101), and (8.102) constitute a coupled autonomous system of differential equations for $\eta_1(\tilde{t})$, $\eta_2(\tilde{t})$, $\eta_3(\tilde{t})$. This system is clearly intimately related to the well-known Whitham modulation equations occurring, for example, in the analysis of the zero dispersion limit of the KdV equation (see [13, 20]).

Now from (8.95)–(8.97),

$$\eta_{1,t} > 0, \tag{8.103}$$

$$\eta_{2,t} < 0, \tag{8.104}$$

$$\eta_{3,t} > 0. \tag{8.105}$$

Thus, as we decrease t, the difference $\eta_2 - \eta_1$ is increasing, and so we may solve (8.62)–(8.64) until one or both of $\eta_1 = -1$, $\eta_2 = \eta_3$ should occur (recall Remark 8.91). We shall now show that in fact the breakdown occurs when $\eta_2 = \eta_3$. More precisely, we make the following

CLAIM. *For $25/6 < t < t_{+,3}^{(1)}$, the solution to the maximization problem is given by $\psi = Re\, F_+$, with F defined in* (8.48), *and the parameters $-1 < \eta_1 < \eta_2 < \eta_3 < 1$ determined by the system* (8.62-8.64). *At $t = 25/6$, we have $\eta_3 = \eta_2$, and $\eta_1 > -1$.*

Define

$$\hat{S} = \left\{ 0 < s < t^{(1)}_{+,3} : \begin{array}{c} \text{for all } t \in (s, t^{(1)}_{+,3}) \text{ there exists} \\ -1 < \eta_1(t) < \eta_2(t) < \eta_3(t) < 1 \text{ solving} \\ \text{(8.62)–(8.64), and the corresponding } \psi \\ \text{solves the maximization problem.} \end{array} \right\}. \tag{8.106}$$

It is clear that $\hat{S}$ is a nonempty left-closed interval, $\hat{S} = [\tau, t^{(1)}_{+,3})$. We have shown (Lemma 6.2) that for t sufficiently small we have $J = (-1, 1)$, so it is also clear that $\tau > 0$. Furthermore, as $\{\eta_j(t)\}_{j=1}^{3}$ are monotone functions of t (see (8.103-8.105) above), we see that

$$\eta_1(t) \downarrow \eta_1(\tau), \tag{8.107}$$

$$\eta_2(t) \uparrow \eta_2(\tau), \tag{8.108}$$

$$\eta_3(t) \downarrow \eta_3(\tau) \tag{8.109}$$

as $t \downarrow \tau$.

We will show first that $\eta_1(\tau) > -1$. Indeed, the quantity T_0 may be evaluated explicitly by contour integration, and setting $\eta_1 = -1$, we obtain

$$T_0 = \frac{-3t}{16} \left[(\eta_2 + \eta_3)^2 + 2(\eta_2 - 1)^2 + 2(\eta_3 - 1)^2 \right], \tag{8.110}$$

which clearly can never vanish.

As we have noted above (see Remark 8.91), whenever $\{\eta_j(t)\}_{j=1}^{3}$, $-1 < \eta_1 < \eta_2 < \eta_3 < 1$, solve (8.62)–(8.64), then necessarily ψ satisfies the variational conditions (8.88)–(8.90). It follows from the definition of τ that we must have $-1 < \eta_1(\tau) < \eta_2(\tau) = \eta_3(\tau) < 1$. Thus for $t = \tau$, the support J of the maximizer ψ is a single interval, i.e. $J = (\eta_1(\tau), 1)$, and ψ vanishes *within* the support,

$$\psi(\eta_2(\tau)) = 0. \tag{8.111}$$

As we know that for $t = \tau$ the support is a single interval, we may use the simpler representation from above, i.e. $\psi = Re\, F_+$, where now F is given by (8.12), (8.13), (8.14). Recall that in this case we have the useful representation (8.21) and (8.22) for ψ and $H\psi$ respectively.

Thus, from (8.14) and (8.21), for $t = \tau$, Eq. (8.111), together with $\psi(z) \geqslant 0$ on $(\eta_1(\tau), 1)$, implies that h must have a double root at $z = \eta_2(\tau) = \eta_3(\tau)$. Hence from (8.14), we must have $\eta_1(\tau) = -3/5$, and $\eta_2(\tau) = \eta_3(\tau) = 2/5$. Also, from (8.17), $\tau = 25/6$. This proves the Claim.

Set $t^{(2)}_{+,3} = 25/6$ and consider $4/3 < t < t^{(2)}_{+,3}$. By elementary calculus (use (8.17) and cf. Fig. 9), we may choose a unique $-1 < \eta = \eta(t) < -3/5$ to solve $T(\eta) = 1$; clearly $\eta(t)$ decreases as we decrease t. To see that the corresponding ψ solves the maximization problem, observe first that for $\eta < -3/5$, $h(z)$ in (8.14) possesses no real roots, and hence $h(z) > 0$ for all $z \in \mathbf{R}$. It is now clear from (8.21) and (8.22) that ψ satisfies (8.19) and (8.20), and hence for $4/3 < t < t^{(2)}_{+,3}$, the maximizer is determined by $\psi = Re\, F_+$, with F given by (8.13), (8.14), and $\eta(t)$ chosen to solve $T(\eta) = 1$. Note that $\eta(t) \downarrow -1$ as $t \downarrow 4/3$. Setting $t^{(3)}_{+,3} = 4/3$, we have shown that for $t \in (t^{(3)}_{+,3}, t^{(2)}_{+,3})$, the support is given by $J = (\eta(t), 1)$, with $\eta(t) \downarrow -1$ as $t \downarrow t^{(3)}_{+,3}$.

We now discuss the region $0 < t < 4/3$. For this region, the solution of the maximization problem is obtained by setting $\eta = -1$ in (8.12), and adding on a homogeneous term,

$$F(z) = \left(\frac{z+1}{z-1}\right)^{1/2} \frac{1}{\pi i} \int_{-1}^{1} \frac{-3it}{2\pi}\, y^2 \left(\frac{y-1}{y+1}\right)^{1/2} \frac{dy}{y-z} + \frac{i\gamma/\pi}{(z^2-1)^{1/2}}, \tag{8.112}$$

with γ chosen to satisfy $\int \psi = 1$:

$$\frac{3t}{4} + \gamma = 1. \tag{8.113}$$

Hence for $t < 4/3$, $\gamma > 0$. As in (8.21) we now have

$$\psi(y) = \frac{3it}{2\pi} \left(\frac{y+1}{y-1}\right)_+^{1/2} h(y) + \frac{i\gamma/\pi}{(y^2-1)_+^{1/2}},$$

which is positive for all $y \in (-1, 1)$. Hence, ψ so defined is indeed the solution of the maximization problem for $0 < t < 4/3$. This completes the proof of cases A and B of Theorem 1.60.

It remains to consider cases C_1–C_4, i.e. $V(x) = tx^{2q+1}$ for general $q > 1$. As in the cases C_1 and C_2 for $V(x) = tx^{2q}$, $q > 2$, we present no further details. Once again the proofs of C_1–C_4 in the present context involve no new ideas, and we leave the details to the reader.

ACKNOWLEDGMENTS

The work of the first author was supported in part by NSF Grant DMS-9500867. The work of the second author was supported in part by the DFG. The work of the third author was supported in part by NSF Postdoctoral Fellowship Grant DMS-9508946, and by Princeton University. The third author also thanks the Department of Mathematics at the Ohio State University, where he held a position during the writing of the manuscript.

The authors are particularly grateful to P. Nevai for introducing (one of) us to the problems considered in this paper, and for generous help and encouragement. The authors are also particularly grateful to V. Totik and E. Saff for providing them with an advance copy of their very useful new book [28] on equilibrium measures. Finally, the authors thank many of their colleagues for very useful conversations,particularly W. Van Assche, A. Magnus, F. Marcellan, P. Nevai, P. Sarnak, V. Totik, and X. Zhou. X. Zhou, in particular, provided great assistance in deriving (2.52) and (2.4) for general C^2 potentials.

REFERENCES

1. P. Blecher and A. Its, Asymptotics of orthogonal polynomials and universality in matrix models, preprint 1996.
2. D. Bessis, C. Itzykson, and J. B. Zuber, Quantum field theory techniques in graphical enumeration, *Adv. Appl. Math.* **1** (1980), 109–157.
3. A. Bouted de Monvel, L. A. Pastur, and M. Shcherbina, On the statistical mechanics approach to random matrix theory: Integrated density of states, *J. Statist. Phys.* **79** (1995), 585–611.
4. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw–Hill, New York, 1955.
5. S. B. Damelin and A. B.J. Kuijlaars, The support of the equilibrium measure in the presence of a monomial external field on $[-1, 1]$, preprint, 1997.
6. P. Deift and K. T.-R. McLaughlin, A continuum limit of the Toda lattice, *Mem. Amer. Math. Soc.* **131** (1998), 624.
7. R. DeVore, "The Approximation of Continuous Functions by Positive Linear Operators," Lecture Notes in Mathematics, Vol. 293, Springer-Verlag, Berlin/Heidelberg/New York, 1972.
8. P. Deift, S. Venakides, and X. Zhou, New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems, submitted for publication.
9. P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the mKdV equation, *Ann. Math.* **137** (1993), 295–370.
10. P. Deift and X. Zhou, Asymptotics for the Painleve II equation, *Comm. Pure Appl. Math.* **48** (1995), 277–337.
11. M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, *Math. Z* **17** (1923), 228–249.
12. M. Fekete, Über den transfiniten Durchmesser ebener Punktmengen, *Math. Z.* **32** (1930), 108–114.
13. H. Flaschka, M. G. Forest, and D. W. McLaughlin, Multiphase averaging and the inverse spectral solution of the KdV equation, *Comm. Pure Appl. Math.* **33** (1980), 739–784.
14. C. F. Gauss, Algemeine Lehzsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte, *in* "Gesammelte Werke (V)" pp. 197–242, Goettingen, 1877.
15. Gonchar and Rakhmanov, Equilibrium measure and the distribution of zeros of extremal polynomials, *Math. USSR Sb.* **53** (1986), 119–130.
16. K. Johansson, On fluctuations of eigenvalues of random hermitian matrices, preprint TRITA-MAT-1995-MA-19, Sept. 1995.
17. A. B.J. Kuijlaars and P. D. Dragnev, Equilibrium problems associated with fast decreasing polynomials, preprint, 1997.
18. A. B.J. Kuijlaars and W. Van Assche, A problem of Totik on fast decreasing polynomials, preprint, 1996.

19. N. S. Landkof, “Foundations of Modern Potential Theory,” Springer-Verlag, Berlin, 1972.
20. P. Lax and C. C.D. Levermore, The small dispersion limit of the Korteweg–de Vries equation I, II, III, *Comm. Pure Appl. Math.* **36** (1983), 253–290 571–593 809–829.
21. K. T.-R. McLaughlin, “A Continuum Limit of the Toda Lattice,” Ph.D. thesis, New York University, 1994.
22. M. L. Mehta, “Random Matrices,” 2nd ed., Academic Press, San Diego, 1991.
23. H. N. Mhaskar and A. B. Saff, Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* **285** (1984), 204–234.
24. H. N. Mhaskar and E. B. Saff, Weighted analogues of capacity, transfinite diameter and Chebyshev constant, *Constr. Approx.* **8** (1992), 105–124.
25. P. Nevai and V. Totik, Weighted polynomial inequalities, *Constr. Approx.* **2** (1986), 113–127.
26. L. A. Pastur, Spectral and probabilistic aspects of matrix models, preprint 1995.
27. E. A. Rakhmanov, On asymptotic properties of polynomials orthogonal on the real axis, *Mat. Sb.* **119** (1982), 163–203; English transl. *Math. USSR Sb.* **47** (1984),.
28. E. B. Saff and V. Totik, Logarithmic potentials with external fields, submitted for publication.
29. E. Stein, “Singular Integrals and Differentiability Properties of Functions,” Princeton Univ. Press, Princeton, NJ, 1970.
30. G. Szegő, “Orthogonal Polynomials,” AMS Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1939.
31. G. Szegő, Bemerkungen zur einer Arbeit von Herrn Fekete “Über den transfiniten Durchmesser ebener Punktmengen, *Math. Z.* **21** (1924), 203–208.
32. F. R. Tian, Oscillations of the zero dispersion limit of the Korteweg–de Vries equation, *Comm. Pure Appl. Math.* (1993).
33. V. Totik, Fast decreasing polynomials via potentials, *J. D'Analyse Math.* **62** (1994), 131–154.
34. S. Venakides, The zero dispersion limit of the Korteweg–de Vries equation with nontrivial reflection coefficient, *Comm. Pure Appl. Math.* **38** (1985), 125–155.
35. S. Venakides, The generation of modulated wavetrains in the solution of the Korteweg–de Vries equation, *Comm. Pure Appl. Math.* **38** (1985), 883–909.
36. S. Venakides, The Korteweg–de Vries equation with small dispersion: Higher order Lax–Levermore theory, *Comm. Pure Appl. Math.* **43** (1990), 335–361.
37. O. Wright, “Korteweg–de Vries Zero Dispersion Limit: A Restricted Initial Value Problem,” Ph.D. thesis Princeton University, Princeton, NJ, 1991.